Examples 3 on Log Identities

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0.0. Using $\log_3 6 = 1.631$, find the value of $\log_3 24$.

0.1. Assuming $\boldsymbol{f(x) = \log_a x}$, and $\boldsymbol{n}$ is constant, put each of $\boldsymbol{f(xy)}$, $\boldsymbol{f(\frac{x}{y})}$, and $\boldsymbol{f(x^n)}$ in terms of $\boldsymbol{f(x)}$ and $\boldsymbol{f(y)}$.

1. Assuming $\boldsymbol{a^5b^2 = 1}$, find the value of $\boldsymbol{\log_a a^4b^5}$.

2. Assuming $\boldsymbol{f(x) = 2^{\frac{x}{2}}}$, evaluate $\boldsymbol{f(x)}$ at $\boldsymbol{x = 2\log_2 3 + \log_4 6 - \log_8 2\sqrt{2}}$.

3. Show that $\boldsymbol{a^{\log b} = b^{\log a}}$.

4. Assuming $\boldsymbol{b^x = u}$, $\boldsymbol{b^y = v}$, and $\boldsymbol{b^z = w}$, express $\boldsymbol{\log_{uv^2w^3} u^3v^2w}$ in terms of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

5. Find the ratio $\boldsymbol{a : b : c}$, assuming that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c > 0}$, but $\boldsymbol{c \neq 1}$, and these:

$\boldsymbol{\log_b (2b + c) + \log_b (2b - c) = 2}$

$\boldsymbol{\log_{a+b} c + \log_{a-b} c = (2\log_{a+b} c)(\log_{a-b} c)}$ where $\boldsymbol{c \neq 1}$

Suggestions or Solutions
To the Problems in the Example 0

0. Using $\log_3 6 = 1.631$, find the value of $\log_3 24$.

We have $\mathbf{\log_3 24 = \log_3 3 + \log_3 8 = 1 + 3 \log_3 2}$.
It looks like we don't have the value of $\mathbf{\log_3 2}$, though.

Yes, we do, and it's in $\mathbf{\log_3 6}$. How then can we get it?

We have $\mathbf{\log_3 6 = \log_3 3 + \log_3 2 = 1 + \log_3 2 = 1.631 \Rightarrow \log_3 2 = 0.631}$.

Thus, we get $\mathbf{\log_3 24 = 1 + 3 \log_3 2 = 1 + 3 \cdot 0.631 = 1 + 1.893 = 2.893}$.

1. Assuming $f(x) = \log_a x$, and n is constant, put each of $f(xy)$, $f\left(\frac{x}{y}\right)$, and $f(x^n)$ in terms of $f(x)$ and $f(y)$.

$f(xy) = \log_a xy = \log_a x + \log_a y = f(x) + f(y)$.

So if $f(x) = \log_a x$, we get $f(xy) = f(x) + f(y)$.

For instance, $f(35) = \log_a 35 = \log_a 5 + \log_a 7 = f(5) + f(7)$.

$f\left(\frac{x}{y}\right) = \log_a \frac{x}{y} = \log_a x - \log_a y = f(x) - f(y)$.

So if $f(x) = \log_a x$, we get $f\left(\frac{x}{y}\right) = f(x) - f(y)$.

For instance, $f\left(\frac{2}{5}\right) = \log_a \frac{2}{5} = \log_a 2 - \log_a 5 = f(2) - f(5)$.

$f(x^n) = \log_a x^n = n \log_a x = nf(x)$.

So if $f(x) = \log_a x$, we get $f(x^n) = nf(x)$.

For instance, $f(2^3) = \log_a 2^3 = 3 \log_a 2 = 3f(2)$.

Suggestions or Solutions
To the Problem in the Example 1

Assuming $a^5b^2 = 1$, find the value of $\log_a a^4b^5$.

$a^5b^2 = 1 \Rightarrow \log_a (a^5b^2) = 5 + 2 \log_a b = 0 \Rightarrow \log_a b = -\frac{5}{2}$.

Thus, $\log_a (a^4b^5) = 4 + 5 \log_a b = 4 + 5(-\frac{5}{2}) = \frac{8-25}{2} = -\frac{17}{2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Solving a problem in math, we break the problem into pieces, and then, put the pieces together so that they add up to the solution.

Basically, this problem is made of a log, $\log_a a^4b^5$ and an equality, $a^5b^2 = 1$.
So we probably have to break into pieces the log and the equality.

Breaking up $\log_a a^4b^5$, we can get $\log_a a^4b^5 = \log_a a^4 + \log_a b^5 = 4 + 5 \log_a b$.
So we get $\log_a a^4b^5 = 4 + 5 \log_a b$.
Then, it seems we want to get the value of $\log_a b$. Where do we get the value, though?

We have $a^5b^2 = 1$ to work with.
So we probably can get the value of $\log_a b$ out of the equality given, $a^5b^2 = 1$. How?

Applying $\log_a$ to $a^5b^2 = 1$, we get $\log_a a^5b^2 = \log_a 1 = 0 \Rightarrow \log_a a^5b^2 = 0$.

Next, we get $\log_a a^5b^2 = \log_a a^5 + \log_a b^2 = 5 + 2 \log_a b = 0 \Rightarrow \log_a b = -\frac{5}{2}$.

Now, going back to $\log_a a^4b^5 = 4 + 5 \log_a b$, and do the substitution, we get this:

$\log_a a^4b^5 = 4 + 5 \log_a b = 4 + 5(-\frac{5}{2}) = \frac{8-25}{2} = -\frac{17}{2}$, which is the solution.

Suggestions or Solutions
To the Problem in the Example 2

Assuming $f(x) = 2^{\frac{x}{2}}$, evaluate $f(x)$ at $x = 2\log_2 3 + \log_4 6 - \log_8 2\sqrt{2}$.

First, we can get $\log_4 6 = \log_{2^2} 6 = \frac{1}{2}\log_2 6 = \log_2 \sqrt{6}$, and

$\log_8 2\sqrt{2} = \log_{2^3} 2^{\frac{3}{2}} = \frac{1}{3}\cdot\frac{3}{2}\log_2 2 = \frac{1}{2}\log_2 2 = \log_2 \sqrt{2}$.

So next, we can get $x = 2\log_2 3 + \log_4 6 - \log_8 2 = \log_2 3^2 + \log_2 \sqrt{6} - \log_2 \sqrt{2}$

$= \log_2 \frac{3^2\sqrt{6}}{\sqrt{2}} = \log_2 3^2\sqrt{3} = \log_2 3^{\frac{5}{2}} = \frac{5}{2}\log_2 3$.

Next, setting $f(\frac{5}{2}\log_2 3) = 2^a$, we get $a = \frac{x}{2} = \frac{\frac{5}{2}\log_2 3}{2} = \frac{5}{4}\log_2 3 = \log_2 3^{\frac{5}{4}}$.

So next, setting $A = f(\frac{5}{2}\log_2 3)$, we get $\log_2 A = \log_2 2^a = a = \log_2 3^{\frac{5}{4}} \Rightarrow A = 3^{\frac{5}{4}}$.

If not quite sure of the idea behind the processes above, follow the steps below.

We have a log identity where $\log_{b^m} A^n = \frac{n}{m}\log_b A$.   How?

We have a log identity where $\log_x y = \dfrac{\log_k y}{\log_k x}$.

So we can get $\log_{b^m} A^n = n\log_{b^m} A = n\dfrac{\log_b A}{\log_b b^m} = n\dfrac{\log_b A}{m\log_b b} = n\dfrac{\log_b A}{m} = \frac{n}{m}\log_b A$.

For instance, we can have this: $\log_8 9 = \log_{2^3} 3^2 = \frac{2}{3}\log_2 3$.

So to begin with, we can get these:

$$\log_4 6 = \log_{2^2} 6 = \tfrac{1}{2}\log_2 6 = \log_2 \sqrt{6}.$$

$$\log_8 2\sqrt{2} = \log_{2^3} 2^{\frac{3}{2}} = \tfrac{1}{3}\cdot\tfrac{3}{2}\log_2 2 = \tfrac{1}{2}\log_2 2 = \log_2 \sqrt{2}.$$

So next, simplifying the value given, we get this:

$$2\log_2 3 + \log_4 6 - \log_8 2 = \log_2 3^2 + \log_2 \sqrt{6} - \log_2 \sqrt{2}$$
$$= \log_2 \tfrac{3^2\sqrt{6}}{\sqrt{2}} = \log_2 3^2\sqrt{3} = \log_2 3^{\frac{5}{2}} = \tfrac{5}{2}\log_2 3.$$

Thus, $\mathbf{2\log_2 3 + \log_4 6 - \log_8 2\sqrt{2}}$ is nothing but $\mathbf{\tfrac{5}{2}\log_2 3}$.

So the solution is the value of $\boldsymbol{f(\tfrac{5}{2}\log_2 3)}$.

Let's first, set $\boldsymbol{f(\tfrac{5}{2}\log_2 3) = 2^{a}}$.

Then, we get $2^{\frac{x}{2}} = 2^{a}$, and thus, we get $a = \frac{x}{2} = \frac{\frac{5}{2}\log_2 3}{2} = \frac{5}{4}\log_2 3 = \log_2 3^{\frac{5}{4}}$.

So we get $a = \log_2 3^{\frac{5}{4}}$, and thus, can readily see that the solution is $3^{\frac{5}{4}}$ by another log identity where $Q = b^{\log_b Q}$. How though?

Assuming $x = b^{\log_b Q}$, and taking $\mathbf{\log_b}$ of both sides, we get this:

$$\log_b x = \log_b b^{\log_b Q} = (\log_b Q)(\log_b b) = \log_b Q \Rightarrow \log_b x = \log_b Q \Rightarrow x = Q.$$

What if however, we forget that identity?

We know $a = \log_2 3^{\frac{5}{4}}$, so taking $\mathbf{\log_2}$ of both sides of $\boldsymbol{f(\tfrac{5}{2}\log_2 3) = 2^{a}}$, we get this:

$$\log_2 f(\tfrac{5}{4}\log_2 3) = \log_2 2^{a} = a = \log_2 3^{\frac{5}{4}} \Rightarrow \log_2 f(\tfrac{5}{4}\log_2 3) = \log_2 3^{\frac{5}{4}} \Rightarrow f(\tfrac{5}{4}\log_2 3) = 3^{\frac{5}{4}}$$

Suggestions or Solutions
To the Problem in the Example 3

Show that $a^{\log b} = b^{\log a}$.

$\log a^{\log b} = (\log b)(\log a) = (\log a)(\log b) = \log b^{\log a} \Rightarrow \log a^{\log b} = \log b^{\log a}$

$\Rightarrow a^{\log b} = b^{\log a}$.

If not quite sure of the idea behind the processes above, follow the steps below.

When solving log equations or doing log algebra, if taking logs of both sides of the equation, equality, inequality, etc., we often see the passage through which we can get to the solution.

Now, we know that the equation given has $\log b$ and $\log a$, which are common logs.
So assuming now, $x = a^{\log b}$, and taking the common log of both sides, we get this:

$\log x = \log a^{\log b} = (\log b)(\log a) = (\log a)(\log b) = \log b^{\log a} \Rightarrow \log x = \log b^{\log a}$

$\Rightarrow x = b^{\log a}$.

Thus, we now have $x = a^{\log b}$, and $x = b^{\log a}$.

Therefore, we get $a^{\log b} = b^{\log a}$.

Thus, we can swap the base and antilog when working with an expression in such a form as $m^{\log n}$, where n is the antilog in a common log, and m is the base.

So for instance, if both $(x^2 - y + 2)$ and $(2y + 3z - 1)$ are positive, we can get this:

$(x^2 - y + 2)^{\log(2y+3z-1)} = (2y + 3z - 1)^{\log(x^2-y+2)}$.

For another simple instance, we have $5^{\log 3} = 3^{\log 5}$.

Suggestions or Solutions
To the Problem in the Example 4

Assuming $b^x = u$, $b^y = v$, and $b^z = w$, express $\log_{uv^2w^3} u^3v^2w$ in terms of x, y, and z.

$b^x = u \Leftrightarrow x = \log_b u$, $b^y = v \Leftrightarrow y = \log_b v$, and $b^z = w \Leftrightarrow z = \log_b w$.

$$\log_{uv^2w^3} u^3v^2w = \frac{\log_b u^3v^2w}{\log_b uv^2w^3} = \frac{\log_b u^3 + \log_b v^2 + \log_b w}{\log_b u + \log_b v^2 + \log_b w^3} = \frac{3\log_b u + 2\log_b v + \log_b w}{\log_b u + 2\log_b v + 3\log_b w}$$

So we get $\log_{uv^2w^3} u^3v^2w = \dfrac{3x + 2y + z}{x + 2y + 3z}$.

If not quite sure of the idea behind the processes above, follow the steps below.

We may want to see first, what x, y, and z are so that we can use them to express the log.

They are embedded in the assumption that $b^x = u$, $b^y = v$, and $b^z = w$. So we want to isolate them. How though?

By the definition for logs, we can extract them the way as follows.
$b^x = u \Leftrightarrow x = \log_b u$, $b^y = v \Leftrightarrow y = \log_b v$, and $b^z = w \Leftrightarrow z = \log_b w$.

And next, we want to see how $\log_{uv^2w^3} u^3v^2w$ is made.
So we want to break apart the log given so that we can see how it can be made.
That is to say that we want to get the breakdown on the log so that we can put it in terms of x, y, and z.

Don't just look at x, y, and z themselves only though.

We want to look at these, too: $\log_b u$, $\log_b v$, and $\log_b w$, since we have these:

$x = \log_b u$, $y = \log_b v$, and $z = \log_b w$.

So breaking up the log, we can break it the way below.

$$\log_{uv^2w^3} u^3v^2w = \frac{\log_b u^3v^2w}{\log_b uv^2w^3} = \frac{\log_b u^3 + \log_b v^2 + \log_b w}{\log_b u + \log_b v^2 + \log_b w^3} = \frac{3\log_b u + 2\log_b v + \log_b w}{\log_b u + 2\log_b v + 3\log_b w}$$

And we know $x = \log_b u$, $y = \log_b v$, and $z = \log_b w$.

So we get $\log_{uv^2w^3} u^3v^2w = \dfrac{3x + 2y + z}{x + 2y + 3z}$.

So the key is the identity where $\log_p Q = \dfrac{\log_r Q}{\log_r p}$, and therefore, changing bases is quite similar to changing denominators in fraction arithmetic.

For instance, $\dfrac{3+8}{5} = \dfrac{\frac{3}{7} + \frac{8}{7}}{\frac{5}{7}} = \dfrac{\frac{3}{19} + \frac{8}{19}}{\frac{5}{19}} = \dfrac{\frac{3}{12.9} + \frac{8}{12.9}}{\frac{5}{12.9}} = \ldots$

And thus, we can put $\dfrac{3\log_b u + 2\log_b v + \log_b w}{\log_b u + 2\log_b v + 3\log_b w}$ the way below, too.

$$\frac{3\log_b u + 2\log_b v + \log_b w}{\log_b u + 2\log_b v + 3\log_b w} = \frac{3\frac{\log u}{\log b} + 2\frac{\log v}{\log b} + \frac{\log w}{\log b}}{\frac{\log u}{\log b} + 2\frac{\log v}{\log b} + 3\frac{\log w}{\log b}} = \frac{3\log u + 2\log v + \log w}{\log u + 2\log v + 3\log w}.$$

In short:

$b^x = u \Leftrightarrow x = \log_b u$, $b^y = v \Leftrightarrow y = \log_b v$, and $b^z = w \Leftrightarrow z = \log_b w$.

$$\log_{uv^2w^3} u^3v^2w = \frac{\log_b u^3v^2w}{\log_b uv^2w^3} = \frac{\log_b u^3 + \log_b v^2 + \log_b w}{\log_b u + \log_b v^2 + \log_b w^3} = \frac{3\log_b u + 2\log_b v + \log_b w}{\log_b u + 2\log_b v + 3\log_b w}$$

So we get $\log_{uv^2w^3} u^3v^2w = \dfrac{3x + 2y + z}{x + 2y + 3z}$.

Suggestions or Solutions
To the Problem in the Example 5

Find the ratio $a : b : c$, assuming that $a, b,$ and $c > 0$, but $c \neq 1$, and these:

$\log_b (2b + c) + \log_b (2b - c) = 2$
$\log_{a+b} c + \log_{a-b} c = (2 \log_{a+b} c)(\log_{a-b} c)$ where $c \neq 1$

To begin with, in the first expression, we get this:

$\log_b (2b + c) + \log_b (2b - c) = \log_b \{(2b + c)(2b - c)\} = \log_b (4b^2 - c^2) = 2$
$\Rightarrow 4b^2 - c^2 = b^2 \Rightarrow 3b^2 = c^2.$

Next, on the left hand side in the second expression, we get this:

$$\log_{a+b} c + \log_{a-b} c = \frac{1}{\log_c (a+b)} + \frac{1}{\log_c (a-b)} = \frac{\log_c (a-b) + \log_c (a+b)}{\{\log_c (a+b)\}\{\log_c (a-b)\}}$$

$$= \frac{\log_c (a^2 - b^2)}{\{\log_c (a+b)\}\{\log_c (a-b)\}}.$$

Next, on the right hand side, we get this:

$$(2\log_{a+b} c)(\log_{a-b} c) = \frac{2}{\log_c (a+b)} \cdot \frac{1}{\log_c (a-b)} = \frac{2}{\{\log_c (a+b)\}\{\log_c (a-b)\}}.$$

So we get $(2 \log_{a+b} c)(\log_{a-b} c) = (2 \log_{a+b} c)(\log_{a-b} c) \Rightarrow \log_c (a^2 - b^2) = 2 \Rightarrow a^2 - b^2 = c^2$.

Besides, we have $3b^2 = c^2$.   Thus, $a^2 - b^2 = c^2 = 3b^2 \Rightarrow a^2 = 4b^2$.

And we know $a, b,$ and c are all positive.
So we get $a^2 = 4b^2 \Rightarrow a = 2b$, and $3b^2 = c^2 \Rightarrow c = \sqrt{3}b$.

Therefore, we can see that the ratio is $a : b : c = 2 : 1 : \sqrt{3}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, simplifying the first equation, we get this:

$\mathbf{log}_b\,(2b + c) + \mathbf{log}_b\,(2b - c) = \mathbf{log}_b\,\{(2b + c)(2b - c)\} = \mathbf{log}_b\,(4b^2 - c^2) = 2$.

Thus, we get $\mathbf{log}_b\,(4b^2 - c^2) = 2$.
And by the definition for logs, we get $4b^2 - c^2 = b^2$, so we get $3b^2 = c^2$.

Next, moving on to $\mathbf{log}_{a+b}\,c + \mathbf{log}_{a-b}\,c = (2\,\mathbf{log}_{a+b}\,c)(\mathbf{log}_{a-b}\,c)$, we have $c \neq 1$.

So beginning with the left hand side, we can put it the way below.

$\mathbf{log}_{a+b}\,c + \mathbf{log}_{a-b}\,c = \dfrac{1}{\mathbf{log}_c\,(a+b)} + \dfrac{1}{\mathbf{log}_c\,(a-b)}$. How?

We have a log identity, $\mathbf{log}_x\,y = \dfrac{\mathbf{log}_y\,y}{\mathbf{log}_y\,x} = \dfrac{1}{\mathbf{log}_y\,x}$, where x and $y > 0$ but $\neq 1$.

And next, moving on to the right hand side, $(2\,\mathbf{log}_{a+b}\,c)(\mathbf{log}_{a-b}\,c)$, we get this:

$$(2\,\mathbf{log}_{a+b}\,c)(\mathbf{log}_{a-b}\,c) = \frac{2}{\mathbf{log}_c\,(a+b)} \cdot \frac{1}{\mathbf{log}_c\,(a-b)} = \frac{2}{\{\mathbf{log}_c\,(a+b)\}\{\mathbf{log}_c\,(a-b)\}}.$$

So we get $(2\,\mathbf{log}_{a+b}\,c)(\mathbf{log}_{a-b}\,c) = \mathbf{log}_{a+b}\,c + \mathbf{log}_{a-b}\,c = \dfrac{1}{\mathbf{log}_c\,(a+b)} + \dfrac{1}{\mathbf{log}_c\,(a-b)}$

$$\Rightarrow \frac{1}{\mathbf{log}_c\,a+b} + \frac{1}{\mathbf{log}_c\,a-b} = \frac{2}{(\mathbf{log}_c\,a+b)(\mathbf{log}_c\,a-b)}.$$

Meanwhile,

$\dfrac{1}{\mathbf{log}_c\,(a+b)} + \dfrac{1}{\mathbf{log}_c\,(a-b)} = \dfrac{\mathbf{log}_c\,(a-b) + \mathbf{log}_c\,(a+b)}{\{\mathbf{log}_c\,(a+b)\}\{\mathbf{log}_c\,(a-b)\}}$, and

$$\mathbf{log}_c\,(a-b) + \mathbf{log}_c\,(a+b) = \mathbf{log}_c\,\{(a-b)(a+b)\} = \mathbf{log}_c\,(a^2 - b^2).$$

Thus, we get $\dfrac{1}{\log_c(a+b)}+\dfrac{1}{\log_c(a-b)}=\dfrac{\log_c(a^2-b^2)}{\{\log_c(a+b)\}\{\log_c(a-b)\}}$.

So we get this:

$$\frac{\log_c(a^2-b^2)}{\{\log_c(a+b)\}\{\log_c(a-b)\}}=\frac{2}{\{\log_c a+b)\}\{\log_c(a-b)\}}\Rightarrow \log_c(a^2-b^2)=2\Rightarrow a^2-b^2=c^2.$$

Besides, we have $3b^2 = c^2$. Thus, we get $a^2 - b^2 = c^2 = 3b^2 \Rightarrow a^2 = 4b^2$.

And we know ***a, b,*** and ***c*** are all positive.
So we get $a^2 = 4b^2 \Rightarrow a = 2b$, and $3b^2 = c^2 \Rightarrow c = \sqrt{3}b$.

Therefore, we can see that the ratio is $a : b : c = 2 : 1 : \sqrt{3}$.

Note:

The ratio above indicates relative values and not actual values.
So a is twice b, and c is $\sqrt{3}$ of b.
Thus, if the actual value of b is k, then that of a is $2k$, and that of c is $\sqrt{3}k$ where $k \neq 1$.
Why $k \neq 1$?

That’s only because of the nature of logarithms.

The value of b itself cannot be 1 because it is used as a base in a logarithm, and thus, is not allowed to be 1.

In the problem definiton, c is not allowed to be 1, either.

Thus, b cannot be $\frac{1}{\sqrt{3}}$, either, in this particular case.

In short:

To begin with, in the first expression, we get this:

$\log_b(2b + c) + \log_b(2b - c) = \log_b\{(2b + c)(2b - c)\} = \log_b(4b^2 - c^2) = 2$
$\Rightarrow 4b^2 - c^2 = b^2 \Rightarrow 3b^2 = c^2$.

Next, on the left hand side in the second expression, we get this:

$$\log_{a+b} c + \log_{a-b} c = \frac{1}{\log_c(a+b)} + \frac{1}{\log_c(a-b)} = \frac{\log_c(a-b) + \log_c(a+b)}{\{\log_c(a+b)\}\{\log_c(a-b)\}}$$

$$= \frac{\log_c(a^2 - b^2)}{\{\log_c(a+b)\}\{\log_c(a-b)\}}.$$

Next, on the right hand side, we get this:

$$(2\log_{a+b} c)(\log_{a-b} c) = \frac{2}{\log_c(a+b)} \cdot \frac{1}{\log_c(a-b)} = \frac{2}{\{\log_c(a+b)\}\{\log_c(a-b)\}}.$$

So we get $(2\log_{a+b} c)(\log_{a-b} c) = (2\log_{a+b} c)(\log_{a-b} c) \Rightarrow \log_c(a^2 - b^2) = 2 \Rightarrow a^2 - b^2 = c^2$.

Besides, we have $3b^2 = c^2$.

Thus, $a^2 - b^2 = c^2 = 3b^2 \Rightarrow a^2 = 4b^2$.

And we know a, b, and c are all positive.

So we get $a^2 = 4b^2 \Rightarrow a = 2b$, and $3b^2 = c^2 \Rightarrow c = \sqrt{3}b$.

Therefore, we can see that the ratio is $a : b : c = 2 : 1 : \sqrt{3}$.