Examples 1 in Number Systems and Logs

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Examples 1 in Number Systems and Logs

Note that a small dot $\cdot$ is a multiplication operator unless specified otherwise, e.g. $3 \times 5 = 3 \cdot 5 = 15$.

Note also, that no use of calculator is allowed in this set of examples.

0. Assuming $\log 2 = 0.3010$, and $\log 3 = 0.4771$, and 2^n a 30-digit integer, find n.

1. It is known that 47^{100} has 168 digits. Find the number of digits in 47^{17}.

2. Suppose a is a positive integer, but b is just a positive number. Suppose also, a^{50} has 42 digits, and b^{-50} has 35 zeroes before the first nonzero appears below the decimal point. Find the number of digits above the decimal point in $(ab)^{10}$.

3. Assuming that $\log x^2$ and $\log \frac{1}{x}$ both have the same mantissa, find the antilog x in $\log x = 2 + m$, where $0 \le m < 1$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming log 2 = 0.3010, log 3 = 0.4771, and 2^n is a 30-digit integer, find *n*.

$\mathbf{2^n}$ is a 30-digit integer $\Rightarrow$ $\boldsymbol{n}$ is an integer, too, and the characteristic = 30 – 1 = 29.

So we get $\mathbf{29 \le \log 2^n < 30}$.

And we have $\mathbf{\log 2^n = n \log 2}$, where $\boldsymbol{n}$ is an integer, and $\mathbf{\log 2 = 0.3010}$. So we get

$$\mathbf{29 \le \log 2^n < 30 \Rightarrow 29 \le n \log 2 < 30 \Rightarrow \frac{29}{\log 2} \le n < \frac{30}{\log 2} \Rightarrow 96.3455 < n < 99.6678}.$$

Therefore, $\boldsymbol{n} = \mathbf{97}$, $\mathbf{98}$, or $\mathbf{99}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, if $\mathbf{2^n}$ is an integer, $\boldsymbol{n}$ is an integer, too. And next, $\mathbf{2^n}$ is positive, of course.

So next, assuming $\mathbf{2^n}$ is a $\boldsymbol{d}$-digit integer, we get $\mathbf{\log 2^n = (d - 1) + m}$, where $\mathbf{0 \le m < 1}$.

And the characteristic of $\mathbf{\log 2^n}$ is $\mathbf{d - 1}$.

So if $\mathbf{2^n}$ has 30 digits, the characteristic of $\mathbf{\log 2^n}$ is $\mathbf{30 - 1 = 29}$.

In other words, $\mathbf{29 \le \log 2^n < 30}$. (So $\mathbf{10^{29} \le 2^n < 10^{30}}$, and thus, $\mathbf{2^n}$ is a 30-digit integer.)

And next, we have $\mathbf{\log 2^n = n \log 2}$, too.

So we get $\mathbf{29 \le \log 2^n < 30 \Rightarrow 29 \le n \log 2 < 30 \Rightarrow \frac{29}{\log 2} \le n < \frac{30}{\log 2}}$.

And we have $\mathbf{\log 2 = 0.3010}$, too. And also, we know $\boldsymbol{n}$ is an integer.

So we get $\mathbf{\frac{29}{0.3010} \le n < \frac{30}{0.3010} \Rightarrow 96.3455 < n < 99.6678}$, and thus, $\boldsymbol{n} = \mathbf{97}$, $\mathbf{98}$, or $\mathbf{99}$.

Suggestions or Solutions
To the Problem in the Example 1

It is known that 47^{100} has 168 digit. Find the number of digits in 47^{17}.

$$167 \leq \log 47^{100} < 168 \Rightarrow 1.67 \leq \log 47 < 1.68$$

$$\Rightarrow 17(1.67) \leq \log 47^{17} < 17(1.68) \Rightarrow 28.3900 \leq \log 47^{17} < 28.56.$$

Therefore, 47^{17} has 29 digits.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose A is the integer. What then does the number of digits in A have to do with?

It has much to do with the highest place value in A.

The highest place value in an n-digit integer is 10^{n-1}.

So assuming that 10^8 is the highest place value in A, we can see A is a 9-digit integer.

Thus, finding the highest place value in A, we can get the number of digits in A.

Taking a log of a number, positive, of course, we can get the information on the highest place value in the number, and if the number is an integer, we can get the information on the number of digits in it, too. The information is called the index or the characteristic.

Now, we want to find the number of digits in 47^{17}, which is an integer decimal.

And we have a log identity, where $x \log A = \log A^x$.

Thus, multiplying $\log 47$ by 17, we can find the characteristic of $\log 47^{17}$.

We need though, to get the value of **log 47**, first. How then can we get it?

Actually, we don't have to get the whole thing about **log 47**.

We need only the characteristic of it, since we are after the number of the digits only.

We can find the characteristic by means of the information on another number $\mathbf{47^{100}}$, and the information is that it has 168 digits. So the characteristic of $\mathbf{\log 47^{100}}$ is 167.

Thus, we can set $\mathbf{\log 47^{100} = 167 + \mathit{m}}$, where $\mathbf{0 \le \mathit{m} < 1}$.

So we get $\mathbf{167 \le \log 47^{100} < 168 \Rightarrow 167 \le 100 \log 47 < 168 \Rightarrow 1.67 \le \log 47 < 1.68}$.

Thus, we can now get the characteristic of $\mathbf{\log 47^{17}}$.

And we have a log identity, $\mathbf{\mathit{k} \log \mathit{A} = \log \mathit{A}^{\mathit{k}}}$. So $\mathbf{17 \log 47 = \log 47^{17}}$. Thus, we get

$\mathbf{1.67 \le \log 47 < 1.68 \Rightarrow 1.67(17) \le 17 \log 47 < 1.68(17) \Rightarrow 28.39 \le \log 47^{17} < 28.56}$.

Thus, we can see that the characteristic of $\mathbf{\log 47^{17}}$ is **28**.

So the highest place value in $\mathbf{47^{17}}$ is $\mathbf{10^{28}}$, and therefore, $\mathbf{47^{17}}$ has **29** digits.

In short:

$\mathbf{167 \le \log 47^{100} < 168 \Rightarrow 1.67 \le \log 47 < 1.68}$

$\mathbf{\Rightarrow 17(1.67) \le \log 47^{17} < 17(1.68) \Rightarrow 28.3900 \le \log 47^{17} < 28.56}$.

Therefore, 47^{17} has 29 digits.

Suggestions or Solutions
To the Problem in the Example 2

Suppose *a* is a positive integer, but *b* is just a positive number.
Suppose also, a^{50} has 42 digits, and b^{-50} has 35 zeroes before the first nonzero appears below the decimal point.

Find the number of digits above the decimal point in $(ab)^{10}$.

$41 \le \log a^{50} < 42 \Rightarrow \frac{41}{50} \le \log a < \frac{42}{50} \Rightarrow 0.82 \le \log a < 0.84$.

$-36 \le \log b^{-50} < -35 \Rightarrow \frac{36}{50} \ge \log b > \frac{36}{50} \Rightarrow 0.7 < \log b \le 0.72$.

Thus, $1.52 < \log a + \log b < 1.56 \Rightarrow 15.2 \le \log (ab)^{10} < 15.6 \Rightarrow$ 16 digits.

If not quite sure of the idea behind the processes above, follow the steps below.

The number of digits in a number has to do with the highest place value in it.

If the number is decimal, the characteristic in the common log of the number can give us the highest place value.

Now, we want to find the number of digits above the decimal point in a number $(ab)^{10}$.

To begin with, finding **log *ab***, first, we can readily get the characteristic of $\log (ab)^{10}$.

We have information on a^{50} and b^{-50}.

The first of the two is an integer, and has 42 digits, so the highest place value in it is 10^{41}.

And the other is just a number, and has 35 zeros before the first nonzero appears below the decimal point, so the highest place value in it is 10^{-36}.

In other words, the charateristic of $\mathbf{log}\ \boldsymbol{a}^{\mathbf{50}}$ is **41**, and that of $\mathbf{log}\ \boldsymbol{b}^{\mathbf{-50}}$ is **-36**.

Let's now, begin with $\mathbf{log}\ \boldsymbol{a}^{\mathbf{50}}$.   Then, we can get this:

$$41 \le \log a^{50} < 42 \Rightarrow 41 \le 50 \log a < 42 \Rightarrow \frac{41}{50} \le \log a < \frac{42}{50} \Rightarrow 0.82 \le \log a < 0.84.$$

Let's next, move on to $\mathbf{log}\ \boldsymbol{b}^{\mathbf{-50}}$, where the characteristic is -36.

Note first, that the mantissa of any log is supposed to be *nonnegative*, and it is.
More specifically, if $\boldsymbol{m}$ is a mantissa, we have $\mathbf{0 \le m < 1}$. In this case therefore,

we need to have $\mathbf{-36 \le \log b^{-50} < -35}$ and **not** $-37 \le \log b^{-50} < -36$.

That's because $\mathbf{\log b^{-50} = -36 + m}$, where $\mathbf{0 \le m < 1}$. So we get

$$-36 \le \log b^{-50} < -35 \Rightarrow -36 \le (-50) \log b < -35 \Rightarrow \frac{36}{50} \ge \log b > \frac{36}{50} \Rightarrow 0.7 < \log b \le 0.72.$$

Now, we have $\mathbf{0.82 \le \log a < 0.84}$, and $\mathbf{0.7 < \log b \le 0.72}$. So we get

$$1.52 < \log a + \log b < 1.56 \Rightarrow 1.52 < \log ab < 1.56 \Rightarrow (1.52)10 \le 10 \log ab < 10(1.56)$$

$$\Rightarrow 15.2 \le \log (ab)^{10} < 15.6.$$

Thus, the characteristic is 15, so we can see that the highest place value in $(ab)^{10}$ is $\mathbf{10^{15}}$.

Therefore, it has 16 digits above the decimal point.

In short:

$$41 \le \log a^{50} < 42 \Rightarrow \frac{41}{50} \le \log a < \frac{42}{50} \Rightarrow 0.82 \le \log a < 0.84.$$

$$-36 \le \log b^{-50} < -35 \Rightarrow \frac{36}{50} \ge \log b > \frac{36}{50} \Rightarrow 0.7 < \log b \le 0.72.$$

Thus, $\mathbf{1.52 < \log a + \log b < 1.56 \Rightarrow 15.2 \le \log (ab)^{10} < 15.6}$ $\Rightarrow$ 16 digits.

Suggestions or Solutions
To the Problem in the Example 3

Assuming that $\log x^2$ and $\log \frac{1}{x}$ both have the same mantissa, find the antilog x in $\log x = 2 + m$, where $0 \le m < 1$.

$\log x = 2 + m$, and $0 \le m < 1 \Rightarrow$ (1) $\log x^2 = 4 + 2m$.
$\log x = 2 + m$, and $0 \le m < 1 \Rightarrow$ (2) $\log \frac{1}{x} = -2 - m = -2 - 1 + 1 - m = -3 + (1 - m)$.

Suppose first, $m = 0$. Then, $\log x = 2 \Rightarrow x = 100$.
Suppose next, $m \ne 0$. Then:

For $\log x^2$, we get $0 < 2m < 1 \Rightarrow 0 < m < \frac{1}{2}$.
For $\log \frac{1}{x}$, we get $0 < 1 - m < 1 \Rightarrow -1 < -m < 0 \Rightarrow 0 < m < 1$.
Thus, (both $0 < m < \frac{1}{2}$ and $0 < m < 1$) $\Rightarrow 0 < m < \frac{1}{2}$.

Let's now, begin with the case where $0 < m < \frac{1}{2}$. Then:
From **(1)**, we get $\log x^2 = 4 + 2m$, and from **(2)**, we get $\log \frac{1}{x} = -3 + (1 - m)$.
Then, $2m = 1 - m \Rightarrow 3m = 1 \Rightarrow m = \frac{1}{3} \Rightarrow \log x = 2 + \frac{1}{3} = \frac{7}{3} \Rightarrow x = 10^{\frac{7}{3}}$.
Let's next, move on to the case where $\frac{1}{2} \le m < 1$.

Then first, from **(1)**, we get $1 \le 2m < 2 \Rightarrow 1 \le$ **mantissa** < 2, which is not allowed.

So we cannot take $2m$ for the mantissa of $\log x^2$, and we need to get the proper one.

From **(1)**, $\log x^2 = 4 + 2m = 4 + 1 - 1 + 2m = 5 + (2m - 1)$.

Let's see if $2m - 1$ can be a mantissa.

$\frac{1}{2} \le m < 1 \Rightarrow 1 \le 2m < 2 \Rightarrow 0 \le 2m - 1 < 1$, which is allowed, since $0 \le$ **mantissa** < 1.

Next, we have $\log \frac{1}{x} = -3 + (1 - m)$. Let's see if $(1 - m)$ can be a mantissa.

$\frac{1}{2} \leq \boldsymbol{m} < \mathbf{1} \Rightarrow \mathbf{-1} < \mathbf{-}\boldsymbol{m} \leq -\frac{1}{2} \Rightarrow \mathbf{0} < \mathbf{1} - \boldsymbol{m} \leq \frac{1}{2}$, which is allowed, since $\mathbf{0} \leq \textbf{mantissa} < \mathbf{1}$.

Then, $\mathbf{2}\boldsymbol{m} - \mathbf{1} = \mathbf{1} - \boldsymbol{m} \Rightarrow \mathbf{3}\boldsymbol{m} = \mathbf{2} \Rightarrow \boldsymbol{m} = \frac{2}{3} \Rightarrow \mathbf{log}\ \boldsymbol{x} = \mathbf{2} + \boldsymbol{m} = \mathbf{2} + \frac{2}{3} = \frac{8}{3} \Rightarrow \boldsymbol{x} = \mathbf{10}^{\frac{8}{3}}$.

Therefore, $\boldsymbol{x} = \mathbf{100}$, $\mathbf{10}^{\frac{7}{3}}$, or $\mathbf{10}^{\frac{8}{3}}$.

If not quite sure of the idea behind the processes above, follow the steps below.

The equation above is for ***m*** rather than ***x***, so it is for the mantissa rather than the antilog. That is, finding the mantissa ***m***, we can readily get the value of the antilog ***x***.

Now, we have a condition that mantissas in both $\mathbf{log}\ \boldsymbol{x}^2$ and $\mathbf{log}\,\frac{1}{x}$ are the same.

So first, we may want to extract the mantissas from both $\mathbf{log}\ \boldsymbol{x}^2$ and $\mathbf{log}\,\frac{1}{x}$.

We can do so putting $\mathbf{log}\ \boldsymbol{x}^2$ and $\mathbf{log}\,\frac{1}{x}$ in terms of the characteristic and mantissa.

Let's now, begin with $\mathbf{log}\ \boldsymbol{x}^2$.

We have $\mathbf{log}\ \boldsymbol{x} = \mathbf{2} + \boldsymbol{m}$, where $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$, and we have $\mathbf{log}\ \boldsymbol{x}^2 = \mathbf{2}\ \mathbf{log}\ \boldsymbol{x}$, too.

So we can get $\mathbf{log}\ \boldsymbol{x} = \mathbf{2} + \boldsymbol{m} \Rightarrow \mathbf{2}\ \mathbf{log}\ \boldsymbol{x} = \mathbf{log}\ \boldsymbol{x}^2 = \mathbf{2(2} + \boldsymbol{m}\mathbf{)} = \mathbf{4} + \mathbf{2}\boldsymbol{m} \Rightarrow \mathbf{log}\ \boldsymbol{x}^2 = \mathbf{4} + \mathbf{2}\boldsymbol{m}$.

Next, we have $\mathbf{log}\,\frac{1}{x} = \mathbf{log}\ \boldsymbol{x}^{-1} = \mathbf{(-1)}\ \mathbf{log}\ \boldsymbol{x} = \mathbf{-log}\ \boldsymbol{x}$. So we get

$\mathbf{log}\ \boldsymbol{x} = \mathbf{2} + \boldsymbol{m} \Rightarrow \mathbf{-log}\ \boldsymbol{x} = \mathbf{-(2} + \boldsymbol{m}\mathbf{)} = \mathbf{-2} - \boldsymbol{m} = \mathbf{-2} - \mathbf{1} + \mathbf{1} - \boldsymbol{m} = \mathbf{-3} + \mathbf{(1} - \boldsymbol{m}\mathbf{)}$. Why?

A mantissa cannot be negative, so the mantissa of $\mathbf{-log}\ \boldsymbol{x}$ is not $\mathbf{-}\boldsymbol{m}$ but $\mathbf{1} - \boldsymbol{m}$ because we have this: $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$.

Still not quite sure why the mantissa of $\mathbf{log}\,\frac{1}{x}$, which is $\mathbf{-log}\ \boldsymbol{x}$, is not $-\boldsymbol{m}$ but $\mathbf{(1} - \boldsymbol{m}\mathbf{)}$?

The bottom line is $\mathbf{0 \le mantissa < 1}$, so basically, a mantissa cannot be negative.

Thus, no mantissa is negative no matter what log it may belong to.
So first, we have $\mathbf{0} \le \boldsymbol{m} < \mathbf{1}$, where $\boldsymbol{m}$ is the mantissa of $\mathbf{log}\ \boldsymbol{x}$, which is a log.

Next, since we have $\mathbf{0} \le \boldsymbol{m} < \mathbf{1}$, we get $\mathbf{-1} < \boldsymbol{-m} \le \mathbf{0}$, so $\boldsymbol{-m}$ is negative or 0, and therefore, is not eligible for a mantissa of any log unless it is 0.

Now, we have $\mathbf{-log}\ \boldsymbol{x} = \mathbf{-2} - \boldsymbol{m}$, where $\mathbf{0} \le \boldsymbol{m} < \mathbf{1}$, and $\mathbf{-log}\ \boldsymbol{x}$ is a log.

So $\boldsymbol{-m}$ cannot be the mantissa of $\mathbf{-log}\ \boldsymbol{x}$ because $\boldsymbol{-m}$ is negative or 0 since $\mathbf{-1} < \boldsymbol{-m} \le \mathbf{0}$.

To keep the mantissa nonnegative, we can put the above in such a way as follows.

$\mathbf{-log}\ \boldsymbol{x} = \mathbf{-3} + (\mathbf{1} - \boldsymbol{m})$, which is equivalent to $\mathbf{-2} - \boldsymbol{m}$.

Then, we can see that the mantissa is $\mathbf{1} - \boldsymbol{m}$, because $\mathbf{0} < \mathbf{1} - \boldsymbol{m} \le \mathbf{1}$.
What about the case where $\mathbf{1} - \boldsymbol{m} = \mathbf{1}$, though?

It will be the case where $\boldsymbol{m} = \mathbf{0}$, and we can take care of it separately.

So before we move on, we may want to set up two different cases for $\boldsymbol{m}$.

One is that $\boldsymbol{m} = \mathbf{0}$, and the other is that $\boldsymbol{m} \ne \mathbf{0}$.

First, if $\boldsymbol{m} = \mathbf{0}$, then $\mathbf{log}\ \boldsymbol{x} = \mathbf{2} + \boldsymbol{m} = \mathbf{-2}$, so we can simply get $\boldsymbol{x} = \mathbf{10^2}$ by the definition for logs. So $\boldsymbol{x} = \mathbf{10^2}$ is a solution.

Suppose next, that $\boldsymbol{m} \ne \mathbf{0}$, that is, we have $\mathbf{0} < \boldsymbol{m} < \mathbf{1}$, and not $\mathbf{0} \le \boldsymbol{m} < \mathbf{1}$.

Then first, the mantissa of $\mathbf{log}\ \frac{1}{x} = \mathbf{-log}\ \boldsymbol{x}$ is not $-m$ but $(\mathbf{1} - \boldsymbol{m})$.

Next, in $\mathbf{log}\ \boldsymbol{x}^{\mathbf{2}} = \mathbf{4} + \mathbf{2}\boldsymbol{m}$, the mantissa is $\mathbf{2}\boldsymbol{m}$, which however, looks problematic.

Since $\boldsymbol{m} \ne \mathbf{0}$, we have $\mathbf{0} < \boldsymbol{m} < \mathbf{1}$. So, we get this:

$\mathbf{0} < \boldsymbol{m} < \mathbf{1} \Rightarrow \mathbf{0} < \mathbf{2}\boldsymbol{m} < \mathbf{2}$, which is **not** OK, because we need to have $\mathbf{0} \le$ **mantissa** $< \mathbf{1}$.

Then, we can do this: $\mathbf{0} \le$ **mantissa** $< \mathbf{1} \Rightarrow \mathbf{0} \le \mathbf{2}\boldsymbol{m} < \mathbf{1} \Rightarrow \mathbf{0} \le \boldsymbol{m} < \frac{1}{2}$.

So we can take $\mathbf{2}\boldsymbol{m}$ for the mantissa of $\mathbf{log}\ \boldsymbol{x}^2$ in the case where $\mathbf{0} < \boldsymbol{m} < \frac{1}{2}$.

Why is 0 not included in the interval?

We've already taken care of the case where $\boldsymbol{m} = \mathbf{0}$, and $\boldsymbol{x} = \mathbf{10}^2$ is the solution in the case.

So we still need to take care of the case where $\mathbf{0} < \boldsymbol{m} < \mathbf{1}$.

Thus, we may want to set up again, two different cases for $\boldsymbol{m}$.

One is that $\mathbf{0} < \boldsymbol{m} < \frac{1}{2}$, and the other is that $\frac{1}{2} \le \boldsymbol{m} < \mathbf{1}$.

Let's however, check to see if $(\mathbf{0} < \boldsymbol{m} < \frac{1}{2})$ satisfies the mantissa of $\mathbf{log}\,\frac{1}{x}$, too.

That's because the mantissas of both $\mathbf{log}\ \boldsymbol{x}^2$ and $\mathbf{log}\,\frac{1}{x}$ are the same.

We know that the mantissa of $\mathbf{log}\,\frac{1}{x}$ is $\mathbf{1} - \boldsymbol{m}$. So we want to see if $\mathbf{1} - \boldsymbol{m}$ is OK.

$\mathbf{0} < \boldsymbol{m} < \frac{1}{2} \Rightarrow -\frac{1}{2} < -\boldsymbol{m} < \mathbf{0} \Rightarrow \frac{1}{2} < \mathbf{1} - \boldsymbol{m} < \mathbf{1}$, which is OK, since $\mathbf{0} <$ **mantissa** $< \mathbf{1}$.

Let's now, begin with the case where $\mathbf{0} < \boldsymbol{m} < \frac{1}{2}$.

In $\mathbf{log}\ \boldsymbol{x}^2$, the mantissa is $\mathbf{2}\boldsymbol{m}$, and in $\mathbf{log}\,\frac{1}{x}$, the mantissa is $(\mathbf{1} - \boldsymbol{m})$.

The two mantissas are the same. So we get $\mathbf{2}\boldsymbol{m} = \mathbf{1} - \boldsymbol{m} \Rightarrow \mathbf{3}\boldsymbol{m} = \mathbf{1} \Rightarrow \boldsymbol{m} = \frac{1}{3}$.

We have $\mathbf{log}\ \boldsymbol{x} = \mathbf{2} + \boldsymbol{m}$. So we get $\mathbf{log}\ \boldsymbol{x} = \mathbf{2} + \frac{1}{3} = \frac{7}{3} \Rightarrow \boldsymbol{x} = \mathbf{10}^{\frac{7}{3}}$ by the definition for logs.

Let's next, move on to the case where $\frac{1}{2} \le \boldsymbol{m} < \mathbf{1}$.

Taking care of $\log x^2$ first, we need to find the proper mantissa of it because $2m$ cannot be the mantissa in this case. So finding the proper one, we can put the log this way, too:

$\log x^2 = 4 + 2m = 4 + 1 - 1 + 2m = 5 + (2m - 1)$.

Next, checking to see if $(2m - 1)$ can be the proper mantissa, we get

$\frac{1}{2} \le m < 1 \Rightarrow 1 \le 2m < 2 \Rightarrow 0 \le 2m - 1 < 1$, which is OK, since $0 \le \text{mantissa} < 1$.

Next, we have $\log \frac{1}{x} = -3 + (1 - m)$.

So checking to see if $(1 - m)$ can be a proper mantissa, we get

$\frac{1}{2} \le m < 1 \Rightarrow -1 < -m \le -\frac{1}{2} \Rightarrow 0 < 1 - m \le \frac{1}{2}$, which is OK, since $0 \le \text{mantissa} < 1$.

Since the two mantissas are the same, we get $2m - 1 = 1 - m \Rightarrow 3m = 2 \Rightarrow m = \frac{2}{3}$.

And we have $\log x = 2 + m$.

So $\log x = 2 + \frac{2}{3} = \frac{8}{3} \Rightarrow x = 10^{\frac{8}{3}}$ by the definition for logs.

Thus, $x = 100, 10^{\frac{7}{3}}$, or $10^{\frac{8}{3}}$. For reference, $10^{\frac{7}{3}} \cong 215.443469$, and $10^{\frac{8}{3}} \cong 464.158883$.

In short:

$\log x = 2 + m$, and $0 \le m < 1 \Rightarrow$ (1) $\log x^2 = 4 + 2m$.
$\log x = 2 + m$, and $0 \le m < 1 \Rightarrow$ (2) $\log \frac{1}{x} = -2 - m = -2 - 1 + 1 - m = -3 + (1 - m)$.

Suppose first, $m = 0$. Then, $\log x = 2 \Rightarrow x = 100$.

Suppose next, $m \ne 0$. Then:

For $\log x^2$, we get $0 < 2m < 1 \Rightarrow 0 < m < \frac{1}{2}$.

For $\log \frac{1}{x}$, we get $0 < 1 - m < 1 \Rightarrow -1 < -m < 0 \Rightarrow 0 < m < 1$.

Thus, (both $0 < m < \frac{1}{2}$ and $0 < m < 1$) $\Rightarrow 0 < m < \frac{1}{2}$.

Let's now, begin with the case where $0 < m < \frac{1}{2}$. Then,

from **(1)** above, we get $\log x^2 = 4 + 2m$, and from **(2)**, we get $\log \frac{1}{x} = -3 + (1 - m)$.

Then, $2m = 1 - m \Rightarrow 3m = 1 \Rightarrow m = \frac{1}{3} \Rightarrow \log x = 2 + \frac{1}{3} = \frac{7}{3} \Rightarrow x = 10^{\frac{7}{3}}$.

Let's next, move on to the case where $\frac{1}{2} \leq m < 1$.

Then first, from **(1)**, we get $1 \leq 2m < 2 \Rightarrow 1 \leq \textbf{mantissa} < 2$, which is not allowed.

So we cannot take $2m$ for the mantissa of $\log x^2$, and we need to get the proper one.

From **(1)**, $\log x^2 = 4 + 2m = 4 + 1 - 1 + 2m = 5 + (2m - 1)$.

Let's see if $2m - 1$ can be a mantissa.

$\frac{1}{2} \leq m < 1 \Rightarrow 1 \leq 2m < 2 \Rightarrow 0 \leq 2m - 1 < 1$, which is allowed, since $0 \leq \textbf{mantissa} < 1$.

Next, we have $\log \frac{1}{x} = -3 + (1 - m)$. Let's see if $(1 - m)$ can be a mantissa.

$\frac{1}{2} \leq m < 1 \Rightarrow -1 < -m \leq -\frac{1}{2} \Rightarrow 0 < 1 - m \leq \frac{1}{2}$, which is allowed, since $0 \leq \textbf{mantissa} < 1$.

Then, $2m - 1 = 1 - m \Rightarrow 3m = 2 \Rightarrow m = \frac{2}{3} \Rightarrow \log x = 2 + m = 2 + \frac{2}{3} = \frac{8}{3} \Rightarrow x = 10^{\frac{8}{3}}$.

Therefore, $x = 100$, $10^{\frac{7}{3}}$, or $10^{\frac{8}{3}}$.