Examples 2 in Number Systems and Logs

Click this to start.

Examples 2 in Number Systems and Logs

Note that a small dot $\cdot$ is a multiplication operator unless specified otherwise, e.g. $3 \times 5 = 3{\cdot}5 = 15$.

Note also, that no use of calculator is allowed in this set of examples.
Instead, you can use a tool as follows:

$$\mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A} = \boldsymbol{s} + \boldsymbol{t} \Leftrightarrow \boldsymbol{s} \le \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A} < \boldsymbol{s} + \mathbf{1} \Leftrightarrow \boldsymbol{b}^{\boldsymbol{s}} \le \boldsymbol{A} < \boldsymbol{b}^{\boldsymbol{s}+\mathbf{1}}, \text{ where } \mathbf{0} \le \boldsymbol{t} < \mathbf{1}.$$

In the tool above, ($\boldsymbol{s}$ **+ 1**) is the number of digits in a number to base ***b***, and ***A*** is positive, of course, and is the decimal equivalent of the number to base ***b***.

Now, assuming **log 2 = 0.3010**, and **log 3 = 0.4771**, do the examples below:

0. Find the highest place value in the binary equivalent of a decimal integer $\mathbf{3^{21}}$.

1. Find the highest place value in the decimal equivalent of a 30-digit binary integer.

Suggestions or Solutions
To the Problem in the Example 0

Find the highest place value in the binary equivalent of a decimal integer 3^{21}.
Use $\log 2 = 0.3010$, and **$\log 3 = 0.4771$.**

$$\log_2 3^{21} = 21 \log_2 3 = 21 \cdot \frac{\log 3}{\log 2} = 21 \cdot \frac{0.4771}{0.3010} \cong 33.2860.$$

Therefore, the highest place value is 2^{33}.

If not quite sure of the idea behind the processes above, follow the steps below.

Converting, of course, the decimal integer 3^{21} into its binary equivalent, then finding the number of digits in the equivalent, we can get the highest place value in the equivalent.

Prior to the conversion though, we need to expand (find the value of) the integer 3^{21}, yet the integer expanded is not a number we can convert into the binary equivalent in a reasonable amount of time.

Expanding it, in fact, we get 10460353203, which is obtained by a calculator, of course. That is, using a calculator, we can get: $10460353203 = 3^{21}$.

We don't have to however, get the expansion, that is, we don't need to get 3^{21} expanded. So we don't need a calculator. And we don't need to convert the expansion, either.

That is to say that we can get the number of digits in the binary equivalent with no use of a calculator. We have a tool to work with, and the tool is called a logarithm.
How then can we use such a tool?

Suppose $\boldsymbol{A}$ is the decimal 3^{21}, that is, the decimal equivalent of the binary in which we want to find the highest place value. Then, taking the log of $\boldsymbol{A}$ to base 2, we can get this:

$\log_2 A = s + t$, where s is an integer, and $0 \le t < 1$.

In the expression above, ***s*** can be called the index for a binary, and ***t*** can be called the mantissa for a binary. Then, the highest place value in the binary equivalent of ***A*** is $\mathbf{2}^{s}$.

That is, the highest place value in the binary equivalent of the decimal 3^{21} is $\mathbf{2}^{s}$.

More specifically, if the binary equivalent is an ***n***-digit binary, ***s*** is $\boldsymbol{n-1}$.

That's because the highest place value an ***n***-digit binary is $\mathbf{2}^{n-1}$.

Now, taking a log of $\mathbf{3^{21}}$ to base 2, we get $\mathbf{\log_2 3^{21} = 21 \log_2 3}$.

So putting into the expression above the value of $\mathbf{\log_2 3}$, we can get the value of $\mathbf{\log_2 3^{21}}$, and in turn, the index. And then, we can get the solution.

How then can we get the value of $\mathbf{\log_2 3}$?

We can take advantage of such an identity as follows. $\log_b A = \dfrac{\log_c A}{\log_c b}$.

Taking advantage of it, we can have $\log_2 3 = \frac{\log 3}{\log 2}$. Thus, we get this:

$\mathbf{\log_2 3^{21} = 21 \log_2 3 = 21 \cdot \frac{\log 3}{\log 2} = 21 \cdot \frac{0.4771}{0.3010} \cong 33.2860}$, where 33 is the index for the base two, and 0.2860 is the mantissa for the base two.

Therefore, the highest place value in the binary equivalent of 3^{21} is 2^{33}.

In other words, the binary equivalent has 34 digits.

In short:

$\mathbf{\log_2 3^{21} = 21 \log_2 3 = 21 \cdot \frac{\log 3}{\log 2} = 21 \cdot \frac{0.4771}{0.3010} \cong 33.2860}$.

Therefore, the highest place value is 2^{33}.

Suggestions or Solutions
To the Problem in the Example 1

Find the highest place value in the decimal equivalent of a 30-digit binary integer. Use log 2 = 0.3010 and log 3 = 0.4771.

Suppose that A is the decimal equivalent. Then,

$$\mathbf{29 \le \log_2 A < 30 \Rightarrow 2^{29} \le A < 2^{30} \Rightarrow \log 2^{29} \le \log A < \log 2^{30}}$$

$$\mathbf{\Rightarrow 29 \log 2 \le \log A < 30 \log 2 \Rightarrow 29(0.3010) \le \log A < 30(0.3010)}$$

$$\mathbf{\Rightarrow 8.7290 \le \log A < 9.0300}$$

$\Rightarrow$ The characteristic of A is **8** or **9**.

Therefore, the highest place value in the decimal equivalent is 10^8 or 10^9.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what are logarithms about?

They are about exponents. Not only that though, but also it can give us a way we can take when looking for the highest place value in a number to any base. So the number can be decimal, binary, octal, etc.

Next, the highest place value in an n-digit integer decimal is $\mathbf{10^{n-1}}$.

So for instance, the highest place value in a 2-digit decimal 89 is this: $\mathbf{10^{2-1} = 10^1 = 10}$, and $\mathbf{10^{3-1} = 10^2 = 100}$ is the highest place value in a 3-digit decimal 903.

What then is the highest place value in an n-digit integer to base b?

It is b^{n-1}. For instance, the highest place value in an n-digit integer octal is 8^{n-1}. So for instance, $8^{2-1} = 8^1 = 8$ is the highest place value in a 2-digit octal 37, which is $3{\cdot}8^1 + 7{\cdot}8^0$, and $2^{6-1} = 2^5 = 32$ is the highest place value in a 6-digit binary 110101, which is $1{\cdot}2^5 + 1{\cdot}2^4 + 0{\cdot}2^3 + 1{\cdot}2^2 + 0{\cdot}2^1 + 1{\cdot}2^0$.

Next, assuming A is a decimal positive, and taking a $\log_2$ of A, what do we get?

We can get $\log_2 A = s + t$, where s is an integer can be called the index for the base two, t can be called the mantissa for the base two, and $0 \le t < 1$.

Then, the highest place value in the binary equivalent of A is 2^s.
That is, the highest place value in the binary equivalent of the decimal 3^{21} is 2^s.

More specifically, if the binary equivalent is an n-digit binary, s is $n - 1$.

That's because the highest place value an n-digit binary is 2^{n-1}.

Suppose now, B is the 30-digit integer binary, and A is the decimal equivalent of B.

Then, the highest place value in B is 2^{29}, so the index of $\log_2 A$ is $30 - 1 = 29$.

Thus, we can set $\log_2 A = 29 + t$, where t is the mantissa for the base two, and $0 \le t < 1$.

So we get $29 \le \log_2 A < 30.$ In other words, we get $2^{29} \le A < 2^{30}$.

What we are after solving this problem is however, the highest place value in A, which is a decimal, so the inequality $2^{29} \le A < 2^{30}$ doesn't give us much help unless we expand 2^{29} and 2^{30}. Such expansions will probably take all day without a calculator.

So it looks like we get to solve for A the inequality, $29 \le \log_2 A < 30$.

Then, we want to extract $\log 2$ from $\log_2 A$.   How?

We have such an identity as follows. $\log_x Y = \dfrac{\log_z Y}{\log_z x}$.

So we can set $\log A = \dfrac{\log A}{\log 2}$, where all the logs are common logs where the base is 10.

Then, we get $29 \le \log_2 A < 30 \Rightarrow 29 \le \dfrac{\log A}{\log 2} < 30 \Rightarrow 29 \log 2 \le \log A < 30 \log 2$.

And we have $\log 2 = 0.3010$, too. So?

So we get $29 \log 2 \le \log A < 30 \log 2 \Rightarrow 29(0.3010) \le \log A < 30(0.3010)$

$\Rightarrow 8.7290 \le \log A < 9.0300$.

We know we can set $\log A = s + t$, where s is an integer called the index (characteristic), and $0 \le t < 1$.

So we can set $8.7290 \le s + t < 9.0300$, where s is an integer, and $0 \le t < 1$.

Thus, we get $s = 8$ or 9. And we know the highest place value in A is 10^s.

So the highest place value of A is 10^8 or 10^9.

In other words, the decimal integer A has 9 digits or 10 digits.

In short:

Suppose that A is the decimal equivalent. Then,

$29 \le \log_2 A < 30 \Rightarrow 2^{29} \le A < 2^{30} \Rightarrow \log 2^{29} \le \log A < \log 2^{30}$

$\Rightarrow 29 \log 2 \le \log A < 30 \log 2 \Rightarrow 29(0.3010) \le \log A < 30(0.3010)$

$\Rightarrow 8.7290 \le \log A < 9.0300 \Rightarrow$ The characteristic of A is 8 or 9.

Therefore, the highest place value in the decimal equivalent is 10^8 or 10^9.