Examples 3 on Logarithms

Click this to start.

Examples 3 on Logarithms

In this set of examples, too, we are going to do some practice on basic algebra on logs so that we get more familiar with manipulation of logs. That's because algebra matters.

0. Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c} > \mathbf{0}$, but $\boldsymbol{b} \neq \mathbf{1}$. Then:

0.0. Assuming that $\boldsymbol{a} = \boldsymbol{b}^{\boldsymbol{k}}$ where $\boldsymbol{k} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{c}$, show that $\boldsymbol{a} = \boldsymbol{c}$.

0.1. Assuming that $\mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{a} = \boldsymbol{a}^{\boldsymbol{k}}$ where $\boldsymbol{k} = \frac{\mathbf{log}_{\boldsymbol{b}} \boldsymbol{N}}{\mathbf{log}_{\boldsymbol{b}} \boldsymbol{a}}$ where $\boldsymbol{N} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{c}$, show that $\boldsymbol{a} = \boldsymbol{c}$.

1. Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c} > \mathbf{0}$, but $\boldsymbol{c} \neq \mathbf{1}$.
Then, assuming $\boldsymbol{B} = \mathbf{log}_{\boldsymbol{c}}\, \boldsymbol{b}$ and $\boldsymbol{A} = \mathbf{log}_{\boldsymbol{c}}\, \boldsymbol{a}$, show that $\boldsymbol{a}^{\boldsymbol{B}} = \boldsymbol{b}^{\boldsymbol{A}}$.

2. Let $\boldsymbol{A}$ and $\boldsymbol{b} > \mathbf{0}$, but $\boldsymbol{b} \neq \mathbf{1}$.
Suppose also, $\mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A} = \boldsymbol{s} + \boldsymbol{t}$, where $\boldsymbol{s}$ is an integer called the index, and $\mathbf{0} \leq \boldsymbol{t} < \mathbf{1}$, where $\boldsymbol{t}$ is called the mantissa. Then:

2.0. Assuming that $\boldsymbol{b} = \mathbf{10}$ in $\mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A}$ above, find the highest place value of $\boldsymbol{A}$, and put the value in terms of the index $\boldsymbol{s}$.

2.1. Assuming that $\boldsymbol{X}$ and $\boldsymbol{Y} > \mathbf{0}$, and that $\mathbf{log}\, \boldsymbol{X}$ and $\mathbf{log}\, \boldsymbol{Y}$ have the same mantissa, but have different indices, show that both $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same sequence of digits, but the positions of the decimal points are different.

2.2. Assuming that $\mathbf{log}\, \boldsymbol{X} - \mathbf{log}\, \boldsymbol{Y} = \boldsymbol{k}$ where $\boldsymbol{k}$ is an integer, find the ratio, $\frac{\boldsymbol{X}}{\boldsymbol{Y}}$.

Suggestions or Solutions
To the Problem 0 in the Example 0

Let a, b, and $c > 0$, but $b \neq 1$. Then, assuming $a = b^k$ where $k = \log_b c$, show that $a = c$.

We have $a = b^k \Leftrightarrow k = \log_b a$ by the definition for logs. Therefore, $a = c$.

If not quite sure of the idea behind the processes above, follow the steps below.

This example is nothing but a matter of the definition for logs.
We have the definition for logs as follows.

• Assuming A and b are real and > 0, but $b \neq 1$, we get $A = b^x \Leftrightarrow x = \log_b A$.

Thus, by the definition, we get $a = b^k \Leftrightarrow k = \log_b a$.

And in this problem, we have $a = b^k$ and $k = \log_b c$. So we can see that $a = c$.

And also, we can come to the same conclusion that $a = c$ in another way.

Taking $\log_b$ of each side in the equality, $a = b^k$, we get $\log_b a = \log_b b^k$.

We have this, too: $\log_b b^k = k \log_b b = k$, since $\log_b b = 1$. So we get $\log_b a = k$.

Besides, we are given $k = \log_b c$. So we get $\log_b a = \log_b c$, and we can see that $a = c$.

In fact, the statement, "$a = b^k$ where $k = \log_b c \Rightarrow a = c$." is saying this: $c = b^{\log_b c}$.

We often use it doing log algebra. For instance,

$15 = 3^{\log_3 15}$, and $3.9 = 1.2^{\log_{1.2} 3.9}$, so $15 = 3^{\log_3 15} = 4^{\log_4 15} = 7^{\log_7 15} = \ldots$

Thus in general, assuming that A, b, c, and $d > 0$, but b, c, and d are $\neq 1$, we get this:

$$A = b^{\log_b A} = c^{\log_c A} = d^{\log_d A} = \ldots$$

Suggestions or Solutions
To the Problem 1 in the Example 0

Let a, b, and $c > 0$, but $b \neq 1$.

Then, assuming $\log_b a = a^k$ where $k = \frac{\log_b N}{\log_b a}$ where $N = \log_b c$, show that $a = c$.

To begin with, $k = \frac{\log_b N}{\log_b a} = \log_a N \Rightarrow k = \log_a N \Rightarrow N = a^k$ by the definition for logs.

Next, $\log_b a = a^k \Rightarrow \log_b a = N$, since $N = a^k$.

Also, we have $N = \log_b c$. Thus, we get $\log_b a = \log_b c$. Therefore, $a = c$.

If not quite sure of the idea behind the processes above, follow the steps below.

The expression looks quite complicated. Taking one step at a time though, we will see it can get actually quite simple.

Let's take care of $k = \frac{\log_b N}{\log_b a}$, first.

We have $\frac{\log_b N}{\log_b a} = \log_a N$. So we get $k = \log_a N$.

Thus, we get $N = a^k$ by the definition for logs.

Now, we have $\log_b a = a^k$, and $N = a^k$. So we get $\log_b a = N$.

Besides, we have $N = \log_b c$, too. So we get $\log_b a = \log_b c$.

Therefore, we can see that $a = c$.

Suggestions or Solutions
To the Problem in the Example 1

Let a, b, and $c > 0$, but $c \neq 1$.
Then, assuming $B = \log_c b$, and $A = \log_c a$, show that $a^B = b^A$.

We have $B = \log_c b$, and $A = \log_c a$.

And setting $k = a^B$, we get

$\log_c k = \log_c a^B = B \log_c a = (\log_c b)(\log_c a) = (\log_c b)A = A \log_c b$.

Thus, $\log_c k = A \log_c b = \log_c b^A$.

So we get $k = b^A$. Besides, we have already set $k = a^B$. Therefore, $a^B = b^A$.

If not quite sure of the idea behind the processes above, follow the steps below.

We have an assumption that $B = \log_c b$ and $A = \log_c a$.

Examining the assumption, we can see that b in $\log_c b$ is used in b^A, and that a in $\log_c a$ is used in a^B. Besides, c is used as the base in each log in the assumption.

So taking $\log_c$ of each side of the equality, $a^B = b^A$, we can get the logs in the assumption.

Then, we can see better what we can do in the next step.

So let's begin with taking $\log_c$ of a^B, which is in the left hand side of the equality we have to show.

Setting $k = a^B$ for simplicity, and taking the log of it, we get $\log_c k = \log_c a^B$.

And we know $B \log_c a = \log_c a^B$. So we get $\log_c k = B \log_c a$.

Besides, we have $\boldsymbol{B} = \mathbf{log}_c\, \boldsymbol{b}$, too. So we get $\mathbf{log}_c\, \boldsymbol{k} = (\mathbf{log}_c\, \boldsymbol{b})(\mathbf{log}_c\, \boldsymbol{a})$.

Also, we have $\boldsymbol{A} = \mathbf{log}_c\, \boldsymbol{a}$. Thus, we get

$\mathbf{log}_c\, \boldsymbol{k} = (\mathbf{log}_c\, \boldsymbol{b})(\mathbf{log}_c\, \boldsymbol{a}) = (\mathbf{log}_c\, \boldsymbol{b})\boldsymbol{A} = \boldsymbol{A}(\mathbf{log}_c\, \boldsymbol{b}) = \boldsymbol{A}\ \mathbf{log}_c\, \boldsymbol{b}$.

Therefore, we get $\mathbf{log}_c\, \boldsymbol{k} = \boldsymbol{A}\ \mathbf{log}_c\, \boldsymbol{b}$.

Now, we know $\boldsymbol{A}\ \mathbf{log}_c\, \boldsymbol{b} = \mathbf{log}_c\, \boldsymbol{b}^{\boldsymbol{A}}$. So we get $\mathbf{log}_c\, \boldsymbol{k} = \mathbf{log}_c\, \boldsymbol{b}^{\boldsymbol{A}}$.

Thus, we get $\boldsymbol{k} = \boldsymbol{b}^{\boldsymbol{A}}$. And yet, we have already set $\boldsymbol{k} = \boldsymbol{a}^{\boldsymbol{B}}$ above.

Therefore, we get $\boldsymbol{a}^{\boldsymbol{B}} = \boldsymbol{b}^{\boldsymbol{A}}$.

In sum, we have $\boldsymbol{a}^{\log_c b} = \boldsymbol{b}^{\log_c a}$. So $\boldsymbol{a}$ and $\boldsymbol{b}$ can exchange their positions.

For instance, $3^{\log_2 7} = 7^{\log_2 3}$. Besides, we can have this, too: $3^{\log_5 7} = 7^{\log_5 3}$.

In short:

We have $\boldsymbol{B} = \mathbf{log}_c\, \boldsymbol{b}$, and $\boldsymbol{A} = \mathbf{log}_c\, \boldsymbol{a}$.

And setting $\boldsymbol{k} = \boldsymbol{a}^{\boldsymbol{B}}$, we get

$\mathbf{log}_c\, \boldsymbol{k} = \mathbf{log}_c\, \boldsymbol{a}^{\boldsymbol{B}} = \boldsymbol{B}\ \mathbf{log}_c\, \boldsymbol{a} = (\mathbf{log}_c\, \boldsymbol{b})(\mathbf{log}_c\, \boldsymbol{a}) = (\mathbf{log}_c\, \boldsymbol{b})\boldsymbol{A} = \boldsymbol{A}\ \mathbf{log}_c\, \boldsymbol{b}$.

Thus, $\mathbf{log}_c\, \boldsymbol{k} = \boldsymbol{A}\ \mathbf{log}_c\, \boldsymbol{b} = \mathbf{log}_c\, \boldsymbol{b}^{\boldsymbol{A}}$.

So we get $\boldsymbol{k} = \boldsymbol{b}^{\boldsymbol{A}}$.

Besides, we have already set $\boldsymbol{k} = \boldsymbol{a}^{\boldsymbol{B}}$.

Therefore, $\boldsymbol{a}^{\boldsymbol{B}} = \boldsymbol{b}^{\boldsymbol{A}}$.

Suggestions or Solutions
To the Problem 0 in the Example 2

Suppose that A and $b > 0$, but $b \neq 1$, and that $\log_b A = s + t$, where s is an integer called the index, and $0 \leq t < 1$, where t is called the mantissa.
Then, assuming that $b = 10$ in $\log_b A$ above, find the highest place value in A, and put the value in terms of the index s.

By the definition for logs, $\log_b A = s + t \Leftrightarrow A = b^{s+t}$, where s is an integer, and $0 \leq t < 1$.

Since $b = 10$, we can set $\log A = s + t \Leftrightarrow A = 10^{s+t}$.

Thus, we have $A = 10^{s+t} = 10^t 10^s$. So we get
$0 \leq t < 1 \Rightarrow 10^0 \leq 10^t < 10^1 \Rightarrow 1 \leq 10^t < 10 \Rightarrow 10^s \leq 10^t 10^s < 10^{s+1} \Rightarrow 10^s \leq A < 10^{s+1}$.

Therefore, the highest place value in A is 10^s.

If not quite sure of the idea behind the processes above, follow the steps below.

When a base $b = 10$ in a log, we usually omit the base.

In that case, such a *log* is called a *common log*, the index is often called a characteristic, and also, we usually use a letter c for indicating the characteristic, and a letter m as the mantissa. Taking a common log of a number A positive, we normally set

$\log A = c + m$, where c is an integer, often called a characteristic, and $0 \leq m < 1$.

Then, 10^c is the highest place value in the number A.

Now, in this example, we are going to see how the highest place value in a number is 10^c.
And math begins with definitions.

So doing problems in math, too, we may want to begin with definitions involved.

Thus, doing this problem, also, we may want to begin with $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$, which is the definition for logs. Of course, $\boldsymbol{x}$ is real, and $\boldsymbol{A}$ and $\boldsymbol{b}$ both are > $\mathbf{0}$, but $\boldsymbol{b \neq 1}$.

Besides, in this problem, even if $\boldsymbol{b = 10}$, we will use $\boldsymbol{s}$ instead of $\boldsymbol{c}$.

That's simply because of consistency, and $\boldsymbol{s}$ is used in the problem description.

And for the same reason, we will use $\boldsymbol{t}$ as the mantissa.

So by the definition for logs, we can set
$\boldsymbol{\log_b A = s + t \Leftrightarrow A = b^{s+t}}$, where $\boldsymbol{s}$ is an integer and $\boldsymbol{0 \leq t < 1}$.

Now, in this problem, we have $\boldsymbol{b = 10}$, so we get $\boldsymbol{\log A = s + t \Leftrightarrow A = 10^{s+t}}$.

And we have $\boldsymbol{0 \leq t < 1}$, too.

Thus, we get $\boldsymbol{0 \leq t < 1 \Rightarrow 10^0 \leq 10^t < 10^1 \Rightarrow 1 \leq 10^t < 10}$.

Next, we have $\boldsymbol{A = 10^{s+t}}$, too, so we get $\boldsymbol{A = 10^t 10^s}$.

Thus, we get $\boldsymbol{1 \leq 10^t < 10 \Rightarrow 10^s \leq 10^t 10^s < 10^{s+1} \Rightarrow 10^s \leq A < 10^{s+1}}$.

Therefore, we can see that $\boldsymbol{10^s}$ is the highest place value in the number $\boldsymbol{A}$. Why?

Suppose $\boldsymbol{w = 10^t}$. Then, since $\boldsymbol{1 \leq 10^t < 10}$, we get $\boldsymbol{1 \leq w < 10}$, so $\boldsymbol{w}$ can be 3.89 for instance.
Of course, $\boldsymbol{w}$ can be 9.89, too. That is, $\boldsymbol{w}$ can be any number ≥ $\mathbf{1}$ and < $\mathbf{10}$.

So we get $\boldsymbol{A = 10^{s+t} = 10^t 10^s = w \cdot 10^s}$, which is often called the scientific notation for numbers, and thus, the highest place value in the antilog $\boldsymbol{A}$ is $\boldsymbol{10^s}$. Still not quite sure?

Suppose for instance, $\boldsymbol{w = 1.5839}$, and $\boldsymbol{s = 3}$.

Then, $\boldsymbol{A} = \mathbf{1.5839 \cdot 10^3 = 1583.9}$, so the highest place value in $\boldsymbol{A}$ is 1000, which is 10^3.

Suppose for another, $\boldsymbol{w} = \mathbf{1.5839}$, and $\boldsymbol{s} = \mathbf{0}$.

Then, $\boldsymbol{A} = \mathbf{1.5839 \cdot 10^0 = 1.5839}$, so the highest place value in $\boldsymbol{A}$ is 1, which is 10^0.

Suppose for one more, $\boldsymbol{w} = \mathbf{3.89}$, and $\boldsymbol{s} = \mathbf{-3}$. Then, $\boldsymbol{A} = \mathbf{3.89 \cdot 10^{-3} = 3.89 \cdot 0.001 = 0.00389}$.

So the highest place value in $\boldsymbol{A}$ is 0.001, which is 10^{-3}.

Therefore:

If the index $\boldsymbol{s} \geq \mathbf{0}$, then the antilog $\boldsymbol{A}$ has $\boldsymbol{s} + \mathbf{1}$ digits above the decimal point.

If the index $\boldsymbol{s} < \mathbf{0}$, then the first nonzero appears in the $\boldsymbol{s}^{\mathbf{th}}$ digit below the decimal point.

In short:

By the definition for logs, $\mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A} = \boldsymbol{s} + \boldsymbol{t} \Leftrightarrow \boldsymbol{A} = \boldsymbol{b}^{\boldsymbol{s}+\boldsymbol{t}}$, where $\boldsymbol{s}$ is an integer, and $\mathbf{0} \leq \boldsymbol{t} < \mathbf{1}$.

Since $\boldsymbol{b} = \mathbf{10}$, we can set $\mathbf{log}\, \boldsymbol{A} = \boldsymbol{s} + \boldsymbol{t} \Leftrightarrow \boldsymbol{A} = \mathbf{10}^{\boldsymbol{s}+\boldsymbol{t}}$.

Thus, we have $\boldsymbol{A} = \mathbf{10}^{\boldsymbol{s}+\boldsymbol{t}} = \mathbf{10}^{\boldsymbol{t}}\mathbf{10}^{\boldsymbol{s}}$.   So we get

$\mathbf{0} \leq \boldsymbol{t} < \mathbf{1} \Rightarrow \mathbf{10^0} \leq \mathbf{10}^{\boldsymbol{t}} < \mathbf{10^1} \Rightarrow \mathbf{1} \leq \mathbf{10}^{\boldsymbol{t}} < \mathbf{10} \Rightarrow \mathbf{10}^{\boldsymbol{s}} \leq \mathbf{10}^{\boldsymbol{t}}\mathbf{10}^{\boldsymbol{s}} < \mathbf{10}^{\boldsymbol{s}+\mathbf{1}} \Rightarrow \mathbf{10}^{\boldsymbol{s}} \leq \boldsymbol{A} < \mathbf{10}^{\boldsymbol{s}+\mathbf{1}}$.

Therefore, the highest place value in $\boldsymbol{A}$ is $\mathbf{10}^{\boldsymbol{s}}$.

Suggestions or Solutions
To the Problem 1 in the Example 2

Suppose that A and $b > 0$, but $b \neq 1$, and that $\log_b A = s + t$, where s is an integer called the index, and $0 \leq t < 1$, where t is called the mantissa.

Then, assuming that X and $Y > 0$, and that $\log X$ and $\log Y$ have the same mantissa, but have different indices, show that both X and Y have the same sequence of digits, but the positions of the decimal points are different.

Since both mantissas are the same, we can set

$\log X = s + m$, and $\log Y = u + m$, where s and u are integers, and $0 \leq m < 1$.

Then, by the definition for logs, we get

$\log X = s + m \Leftrightarrow X = 10^{s+m}$, and $\log Y = u + m \Leftrightarrow Y = 10^{u+m}$.

Next, setting $w = 10^m$, we get

$X = 10^{s+m} = 10^m 10^s = w \cdot 10^s$, and $Y = 10^{u+m} = 10^m 10^u = w \cdot 10^u$.

Next, we have $0 \leq m < 1$, and $w = 10^m$. So we get

$10^0 \leq 10^m < 10^1 \Rightarrow 1 \leq 10^m < 10 \Rightarrow 1 \leq w < 10$.

Thus, we get $X = w \cdot 10^s$, $Y = w \cdot 10^u$, and $1 \leq w < 10$.

Now, w shows the sequence of digits in each of the antilogs X and Y.

Also, u and s can respectively indicate the positions of the decimal points in the antilogs X and Y.

Therefore, X and Y both have the same sequence of digits, but the positions of the decimal points are different.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose first, that:

$\mathbf{log}\ \boldsymbol{X = s + t}$, where $\boldsymbol{s}$ is an integer, and $\mathbf{0 \le \boldsymbol{t} < 1}$.
$\mathbf{log}\ \boldsymbol{Y = u + v}$, where $\boldsymbol{u}$ is an integer, and $\mathbf{0 \le \boldsymbol{v} < 1}$.

Then, $\boldsymbol{s}$ and $\boldsymbol{u}$ are indices, $\boldsymbol{t}$ and $\boldsymbol{v}$ are mantissas, and by the definition for logs, we get
$\mathbf{log}\ \boldsymbol{X = s + t \Leftrightarrow X} = \mathbf{10}^{\boldsymbol{s+t}}$, and $\mathbf{log}\ \boldsymbol{Y = u + v \Leftrightarrow Y} = \mathbf{10}^{\boldsymbol{u+v}}$.

Next, since the mantissas are to be the same, we can set $\boldsymbol{k = t = v}$.

Then, we get $\boldsymbol{X} = \mathbf{10}^{\boldsymbol{s+k}}$, and $\boldsymbol{Y} = \mathbf{10}^{\boldsymbol{u+k}}$, and thus, $\boldsymbol{X} = \mathbf{10}^{\boldsymbol{k}}\mathbf{10}^{\boldsymbol{s}}$, and $\boldsymbol{Y} = \mathbf{10}^{\boldsymbol{k}}\mathbf{10}^{\boldsymbol{u}}$.

Now, we have $\mathbf{0 \le \boldsymbol{t} < 1}$, and $\mathbf{0 \le \boldsymbol{v} < 1}$. Thus, we get $\mathbf{0 \le \boldsymbol{k} < 1}$, since $\boldsymbol{k = t = w}$.

So we get $\mathbf{0 \le \boldsymbol{k} < 1 \Rightarrow 10^0 \le 10^{\boldsymbol{k}} < 10^1 \Rightarrow 1 \le 10^{\boldsymbol{k}} < 10}$.

Suppose next, that $\boldsymbol{w} = \mathbf{10}^{\boldsymbol{k}}$. Then, we get $\mathbf{1 \le \boldsymbol{w} < 10}$, and thus, we get

$\boldsymbol{X} = \mathbf{10}^{\boldsymbol{k}}\mathbf{10}^{\boldsymbol{s}} = \boldsymbol{w}\cdot\mathbf{10}^{\boldsymbol{s}}$, and $\boldsymbol{Y} = \mathbf{10}^{\boldsymbol{k}}\mathbf{10}^{\boldsymbol{u}} = \boldsymbol{w}\cdot\mathbf{10}^{\boldsymbol{u}}$.

Now, $\boldsymbol{w}$ can be any number greater than or equal to 1 and less than 10 such as 3.89053, so $\boldsymbol{w}$ shows the sequence of digits in each of the two antilogs $\boldsymbol{X}$ and $\boldsymbol{Y}$.

That is, $\boldsymbol{X}$ and $\boldsymbol{Y}$ both have the same sequence of digits.

Next, the index $\boldsymbol{u}$ indicates the position of the decimal point in the antilog $\boldsymbol{X}$.
Also, the other index $\boldsymbol{s}$ indicates the position of the decimal point in the antilog $\boldsymbol{Y}$.

For instance, if $\boldsymbol{s} = \mathbf{-2}$, and $\boldsymbol{u} = \mathbf{3}$, we get $\boldsymbol{X} = \mathbf{0.0389053}$, and $\boldsymbol{Y} = \mathbf{3890.53}$.

Therefore, $\boldsymbol{X}$ and $\boldsymbol{Y}$ both have the same sequence of digits, but the positions of the decimal points are different.

Suggestions or Solutions
To the Problem 2 in the Example 2

Suppose that A and $b > 0$, but $b \neq 1$, and that $\log_b A = s + t$, where s is an integer called the index, and $0 \leq t < 1$, where t is called the mantissa.

Then, assuming that $\log X - \log Y = k$ where k is an integer, find the ratio, $\frac{X}{Y}$.

$\mathbf{\log X - \log Y = \log \frac{X}{Y} = k \Leftrightarrow \frac{Y}{X} = 10^k}$ by the definition for logs.

Therefore, the ratio $\mathbf{r = \frac{Y}{X} = 10^k}$.

If not quite sure of the idea behind the processes above, follow the steps below.

We have a log identity where $\mathbf{\log_b A - \log_b C = \log_b \frac{A}{C}}$.
And taking a common log, we usually omit the base 10.

Then, we get $\mathbf{\log X - \log Y = \log \frac{X}{Y}}$.

And we have $\mathbf{\log X - \log Y = k}$, too.

So we get $\mathbf{\log \frac{X}{Y} = k}$.

Then, by the definition for logs, we can get $\mathbf{\log \frac{X}{Y} = k \Leftrightarrow \frac{Y}{X} = 10^k}$.

Therefore, the ratio $\mathbf{\frac{X}{Y} = 10^k}$.

So what else can we mean by the fact that $\mathbf{\log X - \log Y = k}$ where $\mathbf{k}$ is an integer?

Both $\mathbf{\log X}$ and $\mathbf{\log Y}$ have the same mantissa.

That is, for instance, $\mathbf{\log X = 3.1234}$, and $\mathbf{\log Y = 2.1234}$.

So both antilogs $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same sequence of digits.

That is, for instance, $\boldsymbol{X} = \mathbf{1234.567}$, and $\boldsymbol{Y} = \mathbf{123.4567}$.

In short:

$\mathbf{log}\,\boldsymbol{X} - \mathbf{log}\,\boldsymbol{Y} = \mathbf{log}\,\frac{\boldsymbol{X}}{\boldsymbol{Y}} = \boldsymbol{k} \Leftrightarrow \frac{\boldsymbol{Y}}{\boldsymbol{X}} = \mathbf{10}^{\boldsymbol{k}}$ by the definition for logs.

Therefore, the ratio $\boldsymbol{r} = \frac{\boldsymbol{Y}}{\boldsymbol{X}} = \mathbf{10}^{\boldsymbol{k}}$.