Examples 4 on Logarithms

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In this set of examples, too, we are going to do some practice on basic algebra on logs so that we get more familiar with manipulation of logs. That's because algebra matters.

0. Find the number of digits in the octal equivalent of a decimal $\mathbf{3^{100}}$.
Use **log 2 = 0.3010** and **log 3 = 0.4771**.

1. A radioisotope Ra has a half-life of 1602 years. A half-life is the time a radioisotope takes to lose half its radioactivity through decay. Assuming the radioactivity of Ra is 8% of the original in an ancient tool, find the age of the tool. Use **log 2 = 0.3010**.

2. Suppose on education, we will spend 12.7% of the income to be made this year, and income increases by 15.7% each year. Then, by what percent do we have to increase education budget each year during the next 7 years if we want the budget to be 21% of the income of the year after the next seven years? A log table may be used.

Suggestions or Solutions
To the Problem in the Example 0

Find the number of digits in the octal equivalent of a decimal 3^{100}.
Use log 2 = 0.3010 and log 3 = 0.4771.

$$\log_8 3^{100} = 100\log_8 3 = 100\frac{\log 3}{\log 8} = 100\frac{\log 3}{3\log 2} = \frac{100}{3}\frac{\log 3}{\log 2} = \frac{100}{3}\cdot\frac{0.4771}{0.3010} = \frac{47.71}{0.9030} = \frac{47710}{903} = 52 + \frac{754}{903}.$$

Therefore, the highest place value in the octal equivalent is 8^{52}, so the octal equivalent has 53 digits. In other words, it is a 53-digit octal integer.

If not quite sure of the idea behind the processes above, follow the steps below.

We are given a number decimal, and want to get the number of all the digits in the octal equivalent, which is a number to the base 8. So do we need to convert first, the decimal into the octal equivalent?

No, we don’t want to do that. Why not?

If we do so, it will probably take a lot of time, since the decimal given is a huge number.

Using a calculator, we can get $5.15377520732011331036461129765 62\cdot 10^{47}$, which is in fact, a 48-digit number, which is a humongous number we can even hardly read.

So instead, we may want to use logarithms getting the number of digits.

The number given is $\mathbf{3^{100}}$, which is an integer, which is on the base **10**, often simply called a decimal number or just called a decimal for short.

Since the number given is an integer, the octal equivalent is an integer, too. And the number of digits in an integer has much to do with the highest place value in the integer.

The highest place value in a decimal can be indicated by an integer ***c***, usually called the characteristic of the common log of the decimal, and in fact, the highest place value in a decimal is $\mathbf{10}^{c}$, where ***c*** is the integer called the characteristic.

Taking a common log of ***A*** decimal, we can set $\log A = c + m$, where ***c*** is an integer called the characteristic, ***m*** is called the mantissa, and $0 \le m < 1$.

The decimal ***A*** is an antilog in a log, and thus, is supposed to be positive. If negative, take the absolute value of it, or multiply it by **–1**, and then, take the common log of it.

Suppose now, we want to find the highest place value in a number ***B***, which is not a decimal but a number on a base ***b***, and ***A*** is the decimal equivalent of ***B***.

(If for instance, ***b*** is 2, ***B*** is called a binary number, or just quickly called a binary.)

Then, we can find the highest place value in ***B*** taking the log of ***A*** on the base ***b***. Then, we put it this way: $\log_b A = s + t$, where ***s*** is an integer, and $0 \le t < 1$. So ***t*** is the mantissa. In this case however, ***s*** is called the index instead of the characteristic.

Then, b^s is not the highest place value in the decimal ***A*** but the highest place value in ***B***, which is the number on the base ***b***.

(If it sounds however, totally new to you now, refer to the section **Number Systems and Logs** in the book **Powers and Logarithms**.)

Solving thus, the problem in this example, we begin with taking a $\log_8$ of the integer 3^{100}, and then, get the index. So taking first, the log of it, we get this:

$$\log_8 3^{100} = 100\log_8 3 = 100\frac{\log 3}{\log 8} = 100\frac{\log 3}{3\log 2} = \frac{100}{3}\frac{\log 3}{\log 2} = \frac{100}{3}\cdot\frac{0.4771}{0.3010} = \frac{47.71}{0.9030} = \frac{47710}{903} = 52 + \frac{754}{903}.$$

So the index is **52**, and the mantissa is $\frac{754}{903}$.

Thus, we can see that the highest place value in the octal equivalent is 8^{52}.

Therefore, the octal equivalent has 53 digits. Why?

Expanding a decimal 189, we get $\mathbf{189 = 1{\cdot}10^2 + 8{\cdot}10^1 + 9{\cdot}10^0}$.

Thus, 189 in decimal has 3 digits, and the highest place value in 189 is 10^2.

On the other hand, converting the decimal 189 into the octal equivalent, we get

$\mathbf{189 = 2{\cdot}8^2 + 7{\cdot}8^1 + 5{\cdot}8^0 \Rightarrow 275}$, which is the octal equivalent of the decimal 189.

Thus, 275 in octal has 3 digits, and the highest place value in 275 octal is 8^2.

Let's see now, if taking a $\mathbf{\log_8}$ of 189 works.

$$\log_8 189 = \log_8 64 \cdot \tfrac{189}{64} = \log_8 64 + \log_8 \tfrac{189}{64} = 2 + \log_8 \tfrac{189}{64}.$$

We know $1 \le \frac{189}{64} < 8 \Rightarrow \log_8 1 \le \log_8 \frac{189}{64} < \log_8 8 \Rightarrow 0 \le \log_8 \frac{189}{64} < 1$.

So we can set $\mathbf{\log_8 189 = 2 + \mathit{t}}$, where $\mathbf{0 \le \mathit{t} < 1}$.

Therefore, the index is 2, and thus, the highest place value in the octal equivalent to 189 decimal is $\mathbf{8^2}$, and the octal equivalent has 3 digits.

Meanwhile, taking a common log of 3^{100}, we get

$\mathbf{\log 3^{100} = 100 \log 3 = 100{\cdot}0.4771 = 47.71}$.

So the characteristic is 47, which is telling us that the highest place value in 3^{100} is 10^{47}, and that 3^{100} has 48 digits. In other words, 3^{100} is a 48-digit integer.

In short:

$$\log_8 3^{100} = 100 \log_8 3 = 100 \tfrac{\log 3}{\log 8} = 100 \tfrac{\log 3}{3 \log 2} = \tfrac{100}{3} \tfrac{\log 3}{\log 2} = \tfrac{100}{3} \cdot \tfrac{0.4771}{0.3010} = \tfrac{47.71}{0.9030} = \tfrac{47710}{903} = 52 + \tfrac{754}{903}.$$

Therefore, the highest place value in the octal equivalent is 8^{52}, so the octal equivalent has 53 digits. In other words, it is a 53-digit octal integer.

Suggestions or Solutions
To the Problem in the Example 1

A radioisotope Ra has a half-life of 1602 years. A half-life is the time a radioisotope takes to lose half its radioactivity through decay. Assuming the radioactivity of Ra is 8% of the original in an ancient tool, find the age of the tool. Use log 2 = 0.3010.

Suppose $\boldsymbol{A}$ is the original radioactivity of **Ra**.

Then, after the $\boldsymbol{n}^{\text{th}}$ half-life, the radioactivity is $\frac{A}{2^n}$, and the one at present is 0.08$\boldsymbol{A}$.

Thus, $\frac{A}{2^n} = \mathbf{0.08}A \Rightarrow \mathbf{2}^{-n} = \mathbf{0.08} \;\; \Rightarrow \mathbf{2}^{-(n+3)} = \mathbf{0.01}$.

Then, $\log 2^{-(n+3)} = \log 0.01 = -2 \Rightarrow -(n+3)\log 2 = -2 \Rightarrow n + 3 = \frac{2}{\log 2} \Rightarrow n = \frac{2}{\log 2} - 3$.

$\log 2 = 0.3010 \Rightarrow \;\; n = \frac{2}{\log 2} - 3 = \frac{2}{0.3010} - 3 = \frac{2 - 0.9030}{0.3010} = \frac{1.097}{0.3010} = \frac{1097}{301}$.

Therefore, the age of the tool is $\mathbf{1602}n = \mathbf{1602} \cdot \frac{1097}{301} \cong \mathbf{5838.52}$ years old.

If not quite sure of the idea behind the processes above, follow the steps below.

The solution to this problem is the age of the tool, of course.
Basically though, what do we want to get first, finding the solution?

It is the number of half-lives.
More specifically, it is the number of those taken until the radioactivity of **Ra** decrease to 8% of the original. How can we find the number, then?

We can try keeping track of **Ra**'s decay over some half-lives.

What then do we want to begin with?

We may want to begin with setting up the initial and the final states of **Ra**.

So let's suppose $\boldsymbol{A}$ is the original radioactivity.

Then, the final, that is, the present radioactivity is 8% of $\boldsymbol{A}$, which is $\mathbf{0.08}\boldsymbol{A}$.

Now, let's keep track of **Ra**'s decay over several half-lives.

After the first half-life, the radioactivity is half the original $\boldsymbol{A}$, so we get $\frac{A}{2}$. {$\boldsymbol{A}/2$}

After the second one, it is a half of the half, so we get $\frac{\frac{A}{2}}{2} = \frac{A}{2^2}$. {($\boldsymbol{A}/2)/2 = \boldsymbol{A}/2/2 = \boldsymbol{A}/2^2$}

After the third, it's half the amount above, so we get $\frac{\frac{A}{2^2}}{2} = \frac{A}{2^3}$. {($\boldsymbol{A}/2^2)/2 = \boldsymbol{A}/2/2/2 = \boldsymbol{A}/2^3$}

Thus, after the $\boldsymbol{n}^{\text{th}}$ one, we get $\frac{A}{2^n}$. {$\boldsymbol{A}/2^{\boldsymbol{n}}$}

(In fact, $\boldsymbol{n}$ doesn't have to be an integer. We will see why not shortly.)

The radioactivity at present is $\mathbf{0.08}\boldsymbol{A}$. So we can set $\frac{A}{2^n} = \mathbf{0.08}\boldsymbol{A}$, and thus, we get

$$\frac{A}{2^n} = \mathbf{0.08}A \Rightarrow \frac{1}{2^n} = \mathbf{0.08} \Rightarrow \mathbf{2^{-n} = 0.08 = 0.01 \cdot 8} \Rightarrow \frac{2^{-n}}{8} = \mathbf{0.01} \Rightarrow \mathbf{2^{-(n+3)} = 0.01}.$$

Now, we want to solve for $\boldsymbol{n}$ the equation above, which is an exponential equation

Solving such an equation, what do we get as the solution?

We get the exponent, which is another name for a log.

So since it's an exponential equation, we may want to take logs of both sides.

What log then do we want to take?

We are given **log 2 = 0.3010**, which is a common log. Thus, common logs look the best.

So taking the log of each side, we get $\mathbf{\log 2^{-(n+3)} = \log 0.01}$.

Then, beginning with the right hand side, we get $\mathbf{\log 0.01 = \log 10^{-2} = -2}$.

Next, moving on to the left hand side, we get $\mathbf{\log 2^{-(n+3)} = -(n+3)\log 2}$.

So we get $\mathbf{\log 2^{-(n+3)} = \log 0.01 \Rightarrow -(n+3)\log 2 = -2 \Rightarrow n + 3 = \frac{2}{\log 2} \Rightarrow n = \frac{2}{\log 2} - 3}$.

And we have $\mathbf{\log 2 = 0.3010}$, too.

So we get $\mathbf{n \cong \frac{2}{\log 2} - 3 = \frac{2}{0.3010} - 3 = \frac{2 - 0.9030}{0.3010} = \frac{1.097}{0.3010} = \frac{1097}{301}}$.

Now, ***n*** was thought to be the number of half-lives, but doesn't look like an integer.

It does not have to be an integer. In fact, $\mathbf{n \cong \frac{1097}{301} = 3.644518}$..., so it isn't an integer.

Why not, and why not = sign but ≅ sign?

Usually, values from a log table are not exact but approximate. And we have

$\mathbf{n \cong \frac{1097}{301} = \frac{903+194}{301} = 3 + \frac{194}{301}}$, so after the third half-life, it only takes $\mathbf{\frac{194}{301}}$ of the fourth one to reach 8% of the original.   Still not quite sure?

When keeping track of **Ra**'s decay, we've had these:

After the first, $\frac{A}{2}$, which is $A/2$. After the second, $\frac{\frac{A}{2}}{2} = \frac{A}{2^2}$, which is $(A/2)/2 = A/2/2 = A/2^2$.

After the third, $\frac{\frac{A}{2^2}}{2} = \frac{A}{2^3}$, which is $(A/2^2)/2 = A/2/2/2 = A/2^{1+1+1} = A/2^3$.

Now, *in* the fourth, $\mathbf{(A/2^3)/2^{\frac{194}{301}} = A/2/2/2/2^{\frac{194}{301}} = A/2^{1+1+1+\frac{194}{301}} = A/2^{3+\frac{194}{301}} = \frac{A}{2^{\frac{1097}{301}}}}$.

We know that **Ra** has a half-life of 1602 years. So one full half-life takes 1602 years.

Thus, the age of the tool is $\mathbf{1602}\boldsymbol{n} \cong \mathbf{1602} \cdot \frac{\mathbf{1097}}{\mathbf{301}} \cong \mathbf{5838.52}$ years old.

In short:

Suppose $\boldsymbol{A}$ is the original radioactivity of **Ra**.

Then, after the $\boldsymbol{n}^{\text{th}}$ half-life, the radioactivity is $\frac{A}{2^n}$, and the one at present is $0.08\boldsymbol{A}$.

Thus, $\frac{A}{2^n} = \mathbf{0.08}\boldsymbol{A} \Rightarrow \mathbf{2}^{-n} = \mathbf{0.08} \;\Rightarrow \mathbf{2}^{-(n+3)} = \mathbf{0.01}$.

Then, $\mathbf{log\ 2}^{-(n+3)} = \mathbf{log\ 0.01 = -2} \Rightarrow \mathbf{-(}n\mathbf{+3)\ log\ 2 = -2} \Rightarrow n + 3 = \frac{2}{\log 2} \Rightarrow n = \frac{2}{\log 2} - 3$.

$\mathbf{log\ 2 = 0.3010} \Rightarrow \;\; n = \frac{2}{\log 2} - 3 = \frac{2}{0.3010} - 3 = \frac{2 - 0.9030}{0.3010} = \frac{1.097}{0.3010} = \frac{1097}{301}$.

Therefore, the age of the tool is $\mathbf{1602}\boldsymbol{n} = \mathbf{1602} \cdot \frac{\mathbf{1097}}{\mathbf{301}} \cong \mathbf{5838.52}$ years old.

Suggestions or Solutions
To the Problem in the Example 2

Suppose on education, we will spend 12.7% of the income to be made this year, and income increases by 15.7% each year.
Then, by what percent do we have to increase education budget each year during the next 7 years if we want the budget to be 21% of the income of the year after the next seven years? A log table may be used.

Suppose that R_n is the annual income, and that E_n is the annual spending on education.

Suppose also, $n = 0$ for this year. Then, we get these:

$E_0 = \frac{12.7}{100} R_0 = 0.127R_0 \Rightarrow E_0 = 0.127R_0$.

$R_1 = R_0 + \frac{15.7}{100} R_0 = R_0 + 0.157R_0 = R_0(1 + 0.157)$.

$R_2 = R_1 + 15.7\%$ of $R_1 = R_1 + \frac{15.7}{100} R_1 = R_1 + 0.157R_1 = R_1(1 + 0.157)$
$= R_0(1 + 0.157)^2 = 1.157^2R_0$.

So we get $R_{n-1} = 1.157^{(n-1)}R_0$. Thus, the income in the 8th year is $R_7 = 1.157^7R_0$.

Suppose now, that $A\%$ is the annual increase on education budget. Then, we get these:

$E_1 = E_0 + A\%$ of $E_0 = E_0 + \frac{A}{100} E_0 = E_0(1 + 0.01A)$.

$E_2 = E_1 + A\%$ of $E_1 = E_1 + \frac{A}{100} E_1 = E_1(1 + 0.01A) = E_0(1 + 0.01A)^2$.
Now, set $B = 0.01A$.
Then, we get $E_{n-1} = E_0(1 + B)^{n-1}$. Thus, the budget in the 8th year is $E_7 = E_0(1 + B)^7$.

Since E_7 is 21% of R_7, we get $E_7 = 0.21R_7 \Rightarrow E_0(1 + B)^7 = (0.21)1.157^7R_0$.

We have $E_0 = 0.127R_0$, too.

Thus, $\boldsymbol{E_0(1 + B)^7 = (0.21)1.157^7R_0 \Rightarrow 0.127R_0(1 + B)^7 = (0.21)1.157^7R_0}$

$\boldsymbol{\Rightarrow 0.127(1 + B)^7 = (0.21)1.157^7 \Rightarrow 1.27(1 + B)^7 = (2.1)1.157^7}$.

Taking logs of both sides, we get $\boldsymbol{\log 1.27(1 + B)^7 = \log (2.1)1.157^7}$.

$$\log 1.27 + \log (1 + B)^7 = \log 2.1 + \log 1.157^7$$
$$\Rightarrow \log (1 + B)^7 = \log 2.1 + \log 1.157^7 - \log 1.27$$
$$\Rightarrow 7\log (1 + B) = \log 2.1 + 7\log 1.157 - \log 1.27$$
$$\Rightarrow \log (1 + B) = \log 1.157 + \frac{\log 2.1 - \log 1.27}{7}.$$

Using a log table, we get $\boldsymbol{\log 1.157 = 0.0634}$, $\boldsymbol{\log 2.1 = 0.3222}$, and $\boldsymbol{\log 1.27 = 0.1038}$.

Thus, we get $\boldsymbol{\log (1 + B) = \log 1.157 + \frac{\log 2.1 - \log 1.27}{7} \cong 0.0634 + \frac{0.3222 - 0.1038}{7}}$

$\boldsymbol{= 0.0634 + 0.0312 = 0.0946 \Rightarrow \log (1 + B) \cong 0.0946}$.

Using a log table again, we get $\boldsymbol{1 + B \cong 1.24 \Rightarrow B \cong 0.24}$.

We have set $\boldsymbol{B = 0.01A}$ earlier, so we get $\boldsymbol{0.01A \cong 0.24 \Rightarrow A \cong 24}$.

Therefore, we want to increase the education budget by approximately 24% each year during the next 7 years.

If not quite sure of the idea behind the processes above, follow the steps below.

This example is basically the same as the example 7. The solution to this is though, not the number of half-lives but the amount of a half-life if you will. That is, it is not the number of years but the percentage increase. So in this case, too, we may want to begin with setting the initial and final states of the object under consideration. Unlike the case in the example 7 though, the number of such objects is not one but two. One is income, and the other is the budget, which is education spending. What then is the next?

We can try keeping track of each object.
So first, assuming $\boldsymbol{R_0}$ is the income this year, and $\boldsymbol{E_0}$ is the spending on education this year, we get $\boldsymbol{E_0 = 12.7\%}$ **of** $\boldsymbol{R_0 = \frac{12.7}{100} R_0 = 0.127R_0 \Rightarrow E_0 = 0.127R_0}$.

Next, the annual income increases by 15.7% during the next 7 years. So we get these:

The income in this year, that is, the first year = $\boldsymbol{R_0}$.

The income in the second year
$= \boldsymbol{R_1 = R_0 + 15.7}\%$ of $\boldsymbol{R_0 = R_0 + \frac{15.7}{100} \cdot R_0 = R_0 + 0.157R_0 = R_0(1 + 0.157)}$.

The income in the third year
$= \boldsymbol{R_2 = R_1 + 15.7}\%$ of $\boldsymbol{R_1 = R_1 + \frac{15.7}{100} \cdot R_1 = R_1 + 0.157R_1 = R_1(1 + 0.157)}$.

We have $\boldsymbol{R_1 = R_0(1 + 0.157) = 1.157R_0}$, too.

So we get $\boldsymbol{R_2 = R_1(1 + 0.157) = 1.157R_1 = 1.157 \cdot 1.157R_0 = 1.157^2R_0}$.

Thus, by the same token, the income in the $\boldsymbol{n}^{\text{th}}$ year, we get $\boldsymbol{R_{n-1} = 1.157^{(n-1)}R_0}$.

Now, the year after the next seven years is the 8^{th} year.

So the income in the 8^{th} year is $\boldsymbol{R_7 = 1.157^7R_0}$.

Suppose now, that $\boldsymbol{A}$% is the annual increase on education budget.

We know $\boldsymbol{E_0}$ is the education budget this year. So we get these:

The budget in the second year is $\boldsymbol{E_1 = E_0 + A}\%$ of $\boldsymbol{E_0 = E_0 + \frac{A}{100} E_0 = E_0(1 + 0.01A)}$.

The budget in the third year is
$\boldsymbol{E_2 = E_1 + A}\%$ of $\boldsymbol{E_1 = E_1 + \frac{A}{100} E_1 = E_1(1 + 0.01A) = E_0(1+ 0.01A)(1+ 0.01A)}$.

Thus, the budget in the third year is $\boldsymbol{E_2 = E_0(1 + 0.01A)^2}$.

By the same token, in the $\boldsymbol{n}^{\text{th}}$ year, setting $\boldsymbol{B = 0.01A}$, we get $\boldsymbol{E_{n-1} = E_0(1 + B)^{n-1}}$.

Now, the year after the next seven years is the 8^{th} year.

So the budget in the 8^{th} year is $\boldsymbol{E_7 = E_0(1 + B)^7}$, where $\boldsymbol{E_0 = 0.127R_0}$, and $\boldsymbol{B = 0.01A}$.

Now, we want $\boldsymbol{E_7}$ to be 21% of $\boldsymbol{R_7}$, which is the income in the 8^{th} year, and is $\boldsymbol{1.157^7R_0}$.

So we get $\boldsymbol{E_7 = \frac{21}{100} R_7 = 0.21R_7 \Rightarrow E_7 = 0.21R_7 \Rightarrow E_0(1 + B)^7 = 0.21 \cdot 1.157^7R_0}$.

And we have $\boldsymbol{E_0 = 0.127R_0}$, too.

So we get $\boldsymbol{E_0(1 + B)^7 = 0.21 \cdot 1.157^7R_0 \Rightarrow 0.127R_0(1 + B)^7 = 0.21 \cdot 1.157^7R_0}$.

Canceling out $\boldsymbol{R_0}$, we get $\boldsymbol{0.127(1 + B)^7 = 0.21 \cdot 1.157^7}$.

Let's put it this way, though: Multiplying both sides by 10, we get

$\boldsymbol{1.27(1 + B)^7 = 2.1 \cdot 1.157^7}$, because we are going to use a log table.

Now, we get to solve the above for $\boldsymbol{B}$.
So we want to remove the exponent 7 from $\boldsymbol{(1 + B)^7}$. Exponent is another name for a log.

So taking logs of both sides, we get $\boldsymbol{\log 1.27(1 + B)^7 = \log 2.1 \cdot 1.157^7}$, and we get

$$\boldsymbol{\log 1.27 + \log (1 + B)^7 = \log 2.1 + \log 1.157^7}$$

$$\boldsymbol{\Rightarrow \log (1 + B)^7 = \log 2.1 + \log 1.157^7 - \log 1.27}$$

$$\boldsymbol{\Rightarrow 7 \log (1 + B) = \log 2.1 + 7 \log 1.157 - \log 1.27}$$

$$\boldsymbol{\Rightarrow \log (1 + B) = \log 1.157 + \frac{\log 2.1 - \log 1.27}{7}}.$$

Let's now use a log table.

Then, we get $\mathbf{\log 1.157 = 0.0634}$, $\mathbf{\log 2.1 = 0.3222}$, and $\mathbf{\log 1.27 = 0.1038}$.

Thus, we get $\mathbf{\log (1 + \mathit{B}) = \log 1.157 + \frac{\log 2.1 - \log 1.27}{7} \cong 0.0634 + \frac{0.3222 - 0.1038}{7}}$

$\mathbf{= 0.0634 + 0.0312 = 0.0946 \Rightarrow \log (1 + \mathit{B}) \cong 0.0946}$.

Usually, values from a log table are not exact but approximate, so we want to use ≅ sign and not = sign.

Now, using a log table again, we get $\mathbf{1 + \mathit{B} \cong 1.24}$.

Thus, we get $\mathbf{\mathit{B} \cong 0.24}$. We have set $\mathbf{\mathit{B} = 0.01\mathit{A}}$, earlier.

So we get $\mathbf{0.01\mathit{A} \cong 0.24 \Rightarrow \mathit{A} \cong 24}$.

Therefore, we want to increase the education budget by approximately 24% each year during the next 7 years.

Using a calculator, we get $\mathbf{\log (1 + \mathit{B}) \cong 0.0946 \Rightarrow 1 + \mathit{B} \cong 10^{0.0946} \cong 1.2434}$.

In short:

Suppose that $\boldsymbol{R_n}$ is the annual income, and that $\boldsymbol{E_n}$ is the annual spending on education.

Suppose also, $\boldsymbol{n} = \mathbf{0}$ for this year. Then, we get these:

$\boldsymbol{E_0 = \frac{12.7}{100} R_0 = 0.127 R_0 \Rightarrow E_0 = 0.127 R_0}$.

$\boldsymbol{R_1 = R_0 + \frac{15.7}{100} R_0 = R_0 + 0.157 R_0 = R_0(1 + 0.157)}$.

$\boldsymbol{R_2 = R_1 + 15.7\%}$ of $\boldsymbol{R_1 = R_1 + \frac{15.7}{100} R_1 = R_1 + 0.157 R_1 = R_1(1 + 0.157)}$
$= \boldsymbol{R_0(1 + 0.157)^2} = \boldsymbol{1.157^2 R_0}$.

So we get $\boldsymbol{R_{n-1} = 1.157^{(n-1)} R_0}$. Thus, the income in the 8th year is $\boldsymbol{R_7 = 1.157^7 R_0}$.

Suppose now, that A% is the annual increase on education budget. Then, we get these:

$\boldsymbol{E_1 = E_0 + A}$% of $\boldsymbol{E_0 = E_0 + \frac{A}{100}E_0 = E_0(1 + 0.01A)}$.

$\boldsymbol{E_2 = E_1 + A}$% of $\boldsymbol{E_1 = E_1 + \frac{A}{100}E_1 = E_1(1 + 0.01A) = E_0(1 + 0.01A)^2}$.
Now, set $\boldsymbol{B = 0.01A}$.
Then, we get $\boldsymbol{E_{n-1} = E_0(1 + B)^{n-1}}$. Thus, the budget in the 8th year is $\boldsymbol{E_7 = E_0(1 + B)^7}$.

Since E_7 is 21% of R_7, we get $\boldsymbol{E_7 = 0.21R_7 \Rightarrow E_0(1 + B)^7 = (0.21)1.157^7R_0}$.

We have $\boldsymbol{E_0 = 0.127R_0}$, too.
Thus, $\boldsymbol{E_0(1 + B)^7 = (0.21)1.157^7R_0 \Rightarrow 0.127R_0(1 + B)^7 = (0.21)1.157^7R_0}$
$\boldsymbol{\Rightarrow 0.127(1 + B)^7 = (0.21)1.157^7 \Rightarrow 1.27(1 + B)^7 = (2.1)1.157^7}$.

Taking logs of both sides, we get $\boldsymbol{\log 1.27(1 + B)^7 = \log (2.1)1.157^7}$.

$\boldsymbol{\log 1.27 + \log (1 + B)^7 = \log 2.1 + \log 1.157^7}$
$\boldsymbol{\Rightarrow \log (1 + B)^7 = \log 2.1 + \log 1.157^7 - \log 1.27}$
$\boldsymbol{\Rightarrow 7\log (1 + B) = \log 2.1 + 7\log 1.157 - \log 1.27}$
$\boldsymbol{\Rightarrow \log (1 + B) = \log 1.157 + \frac{\log 2.1 - \log 1.27}{7}}$.

Using a log table, we get $\boldsymbol{\log 1.157 = 0.0634}$, $\boldsymbol{\log 2.1 = 0.3222}$, and $\boldsymbol{\log 1.27 = 0.1038}$.

Thus, we get $\boldsymbol{\log (1 + B) = \log 1.157 + \frac{\log 2.1 - \log 1.27}{7} \cong 0.0634 + \frac{0.3222 - 0.1038}{7}}$
$\boldsymbol{= 0.0634 + 0.0312 = 0.0946 \Rightarrow \log (1 + B) \cong 0.0946}$.

Using a log table again, we get $\boldsymbol{1 + B \cong 1.24 \Rightarrow B \cong 0.24}$.

We have set $\boldsymbol{B = 0.01A}$ earlier, so we get $\boldsymbol{0.01A \cong 0.24 \Rightarrow A \cong 24}$.

Therefore, we want to increase the education budget by approximately 24% each year during the next 7 years.