Examples 5 on Logarithms

Click this to start.

Examples 5 in Logarithms

In this set of examples, too, we are going to do some practice on basic algebra on logs so that we get used to manipulation of logs. That's because, of course, algebra matters.

0.0. Assuming $\log_{16} 7 = x$ and $16^y = 17$, put $\log_{64} 119$ in terms of x and y.

0.1. Assuming $\log_{14} 7 = x$ and $14^y = 8$, put $\log_{28} 56$ in terms of x and y.

1. Show that $(\log_{15} 3)^2 + \dfrac{\log_{15} 45}{\log_5 15} = 1$.

2. Suppose that a, b, and $c > 0$, but that a, b, and $c \neq 1$. Suppose also, that we have these:

p, q, and $r \neq 0$.
$pq + qr + rp = pqr$.
$\log_a t = p$, $\log_b t = q$, and $\log_c t = r$.

Then, put t in terms of a, b, and c.

3. Assuming $1 < c < d$, and $c^{\frac{1}{m}} = d^{\frac{1}{n}} = cd$, show this: $\sqrt{m^3 - m^2 n - mn^2 + n^3} = \log_{dc} \frac{d}{c}$.

Suggestions or Solutions
To the Problem 0 in the Example 0

Assuming $\log_{16} 7 = x$, and $16^y = 17$, put $\log_{64} 119$ in terms of x and y.

$16^y = 17 \Rightarrow \log 16^y = \log 17 \Rightarrow y \log 16 = \log 17 \Rightarrow y = \dfrac{\log 17}{\log 16} = \log_{16} 17.$

$\log_{64} 119 = \dfrac{\log_{16} 119}{\log_{16} 64}$. $\log_{16} 64 = \log_{4^2} 4^3 = \frac{3}{2}\log_4 4 = \frac{3}{2}$. $\log_{16} 119 = \log_{16} 17\cdot 7 = x + y$.

Therefore, $\log_{64} 119 = \frac{x+y}{\frac{3}{2}} = \frac{2}{3}(x+y)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Since we need to put something in terms of x and y, we want to know what x and y are. We have $x = \log_{16} 7$. What about y?

We have $16^y = 17$, which is an exponential equation, so we want to solve it for y.

Solving such an equation as above, we get as the solution an exponent, which is another name for a log. So taking logs of both sides, we get

$\log 16^y = \log 17 \Rightarrow y \log 16 = \log 17 = \dfrac{\log 17}{\log 16}$. Besides, we have $\log_a b = \dfrac{\log_c b}{\log_c a}$, too.

So we get $y = \dfrac{\log 17}{\log 16} = \log_{16} 17$.

Thus, we now have $x = \log_{16} 7$, and $y = \log_{16} 17$, both of which are on the base 16.

So we want to put $\log_{64} 119$ in some logs on the base 16. How?

Using the same identity as above again, we get $\mathbf{\log_{64} 119 = \frac{\log_{16} 119}{\log_{16} 64}}$.

Meanwhile, $\mathbf{\log_{16} 64 = \log_{16} 16 \cdot 4 = \log_{16} 16 + \log_{16} 4 = 1 + \log_{16} 16^{0.5} = 1 + 0.5 = 1.5}$.

Thus, we get $\mathbf{\log_{64} 119 = \frac{\log_{16} 119}{1.5} = \frac{\log_{16} 119}{\frac{3}{2}} = \frac{2}{3}\log_{16} 119}$. Now, we should be able to put 119 in terms of 7 and 17, since we have $\boldsymbol{x} = \mathbf{\log_{16} 7}$, and $\boldsymbol{y} = \mathbf{\log_{16} 17}$.
Factorizing 119, we get $\mathbf{119 = 17 \cdot 7}$.

So we get $\mathbf{\log_{16} 119 = \log_{16} 7 \cdot 17 = \log_{16} 7 + \log_{16} 17}$, which is $\boldsymbol{x + y}$.

Therefore, $\mathbf{\log_{64} 119 = \frac{2}{3}(\boldsymbol{x + y})}$.

In short:

$\mathbf{16^{\boldsymbol{y}} = 17 \Rightarrow \log 16^{\boldsymbol{y}} = \log 17 \Rightarrow \boldsymbol{y} \log 16 = \log 17 \Rightarrow \boldsymbol{y} = \frac{\log 17}{\log 16} = \log_{16} 17}$.

$\mathbf{\log_{64} 119 = \frac{\log_{16} 119}{\log_{16} 64}}$. $\mathbf{\log_{16} 64 = \log_{4^2} 4^3 = \frac{3}{2}\log_4 4 = \frac{3}{2}}$. $\mathbf{\log_{16} 119 = \log_{16} 17 \cdot 7 = \boldsymbol{x + y}}$.

Therefore, $\mathbf{\log_{64} 119 = \frac{\boldsymbol{x+y}}{\frac{3}{2}} = \frac{2}{3}(\boldsymbol{x + y})}$.

Suggestions or Solutions
To the Problem 1 in the Example 0

Assuming $\log_{14} 7 = x$, and $14^y = 8$, put $\log_{28} 56$ in terms of x and y.

To begin with, $\log 14^y = \log 8 \Rightarrow y \log 14 = \log 8 \Rightarrow y = \dfrac{\log 8}{\log 14} = \log_{14} 8.$

Next, $\log_{28} 56 = \dfrac{\log_{14} 56}{\log_{14} 28} = \dfrac{\log_{14} 7 + \log_{14} 8}{\log_{14} 14 + \log_{14} 2} = \dfrac{x + y}{1 + \log_{14} 2}.$

Meanwhile, $\log_{14} 2 = \log_{14} \frac{14}{2} = \log_{14} 14 - \log_{14} 7 = 1 - x.$

Therefore, $\log_{28} 56 = \dfrac{x + y}{1 + \log_{14} 2} = \dfrac{x + y}{1 + 1 - x} = \dfrac{x + y}{2 - x}.$

If not quite sure of the idea behind the processes above, follow the steps below.

Putting something in terms of x and y, we want to know what x and y are.
We have $x = \log_{14} 7$, but not y. So what about y?

We can get y from one of the two equations above, which is $14^y = 8$.
So we want to solve an exponential equation.
Solving such an equation, we often take logs of both sides of the equation.

So taking logs of both sides in the equation $14^y = 8$, we get

$$\log 14^y = \log 8 \Rightarrow y \log 14 = \log 8 \Rightarrow y = \frac{\log 8}{\log 14} = \log_{14} 8.$$

Thus, we now have $x = \log_{14} 7$, and $y = \log_{14} 8$, both of which are on the base 14, so we want to put $\log_{28} 56$ in terms of logs on the base 14.

We have an identity, $\log_a b = \log_a b = \frac{\log_c b}{\log_c a}$, so we can set $\log_{28} 56 = \frac{\log_{14} 56}{\log_{14} 28}$.

We know $x + y = \log_{14} 7 + \log_{14} 8 = \log_{14} 56$, since $56 = 7{\cdot}8$. What then about $\log_{14} 28$?

We have $28 = 14{\cdot}2$, so we get $\log_{14} 28 = 1 + \log_{14} 2$. Well then what about 2?

We have $2 = \frac{14}{7}$, so we get $\log_{14} 2 = \log_{14} \frac{14}{7} = 1 - \log_{14} 7 = 1 - x$, since $x = \log_{14} 7$.

Thus, we get $\log_{14} 28 = 1 + \log_{14} 2 = 1 + (1 - x) = 2 - x$.

Therefore, we can see this: $\log_{28} 56 = \frac{\log_{14} 56}{\log_{14} 28} = \frac{x + y}{2 - x}$.

In short:

To begin with, $\log 14^y = \log 8 \Rightarrow y \log 14 = \log 8 \Rightarrow y = \frac{\log 8}{\log 14} = \log_{14} 8$.

Next, $\log_{28} 56 = \frac{\log_{14} 56}{\log_{14} 28} = \frac{\log_{14} 7 + \log_{14} 8}{\log_{14} 14 + \log_{14} 2} = \frac{x + y}{1 + \log_{14} 2}$.

Meanwhile, $\log_{14} 2 = \log_{14} \frac{14}{2} = \log_{14} 14 - \log_{14} 7 = 1 - x$.

Therefore, $\log_{28} 56 = \frac{x + y}{1 + \log_{14} 2} = \frac{x + y}{1 + 1 - x} = \frac{x + y}{2 - x}$.

Suggestions or Solutions
To the Problem in the Example 1

Show that $(\log_{15} 3)^2 + \dfrac{\log_{15} 45}{\log_5 15} = 1.$

To begin with, $\dfrac{\log_{15} 45}{\log_5 15} = \log_{15} 45 \cdot \dfrac{1}{\log_5 15}$.

Meanwhile, $\dfrac{1}{\log_5 15} = \dfrac{1}{\frac{\log_{15} 15}{\log_{15} 5}} = \dfrac{1}{\frac{1}{\log_{15} 5}} = (\log_{15} 5)$.

So $\dfrac{\log_{15} 45}{\log_5 15} = \log_{15} 45 \cdot \dfrac{1}{\log_5 15} = (\log_{15} 45)(\log_{15} 5)$.

Next, $(\log_{15} 45)(\log_{15} 5) = (\log_{15} 5 + \log_{15} 9)(\log_{15} 5) = (\log_{15} 5 + 2\log_{15} 3)(\log_{15} 5)$

$= (\log_{15} 5)^2 + 2(\log_{15} 3)(\log_{15} 5)$.

Thus, $(\log_{15} 3)^2 + \dfrac{\log_{15} 45}{\log_5 15}$

$= (\log_{15} 3)^2 + (\log_{15} 5)^2 + 2(\log_{15} 3)(\log_{15} 5) = (\log_{15} 3 + \log_{15} 5)^2 = (\log_{15} 15)^2 = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

Consistency matters in math.
To begin with, checking the bases in all the logs in the equality above, we can see that **$\log_5 15$** only has a different base.
So we may want to make it have the base 15, too.   How?

We can apply a log identity, $\log_a b = \frac{1}{\log_b a}$, where both a and $b > 0$, but $\neq 1$, of course.

Then, we get $\mathbf{\log_5 15 = \frac{1}{\log_{15} 5}}$.

So we can put the equality above this way, too: $\mathbf{(\log_{15} 3)^2 + (\log_{15} 45)(\log_{15} 5) = 1}$.

Now, what we are going to do is to force the right side to be 1, so it looks like we've got to do some algebra.

First, the square term, $\mathbf{(\log_{15} 3)^2}$ is **not** $\log_{15} 3^2$, so there seems to be not much we can do about it, and thus, we may want to keep it intact, for now.

Then, it looks like we want to do some modifications to $\mathbf{(\log_{15} 45)(\log_{15} 5)}$.

So we are going to break it into pieces, and then, put the pieces together. We don't just break it, of course, and we don't just put them together, either. Both have to add up. Well, that's basically what we do in math.

Thus, doing the modification, we want to keep in mind that the sum of the modification result and $\mathbf{(\log_{15} 3)^2}$ should add up to 1.

So let's have a closer look at the right hand side.

Closely looking at all the numbers used, we can notice that they are made of 3 and 5.

$\mathbf{15 = 3 \cdot 5}$, and **45** can be factorized to $\mathbf{9 \cdot 5 = 3^2 \cdot 5}$.

We know that the base is 15, and that $\mathbf{\log_{15} 15 = 1}$.

Besides, we know $\mathbf{\log_{15} 3 + \log_{15} 5 = \log_{15} 3 \cdot 5 = \log_{15} 15 = 1}$, too.

In addition, we know $\mathbf{(\log_{15} 3 + \log_{15} 5)^2 = (\log_{15} 15)^2 = 1}$, as well.

Now, expanding $\mathbf{(\log_{15} 3 + \log_{15} 5)^2}$, we get $\mathbf{(\log_{15} 3)^2 + 2(\log_{15} 3)(\log_{15} 5) + (\log_{15} 5)^2}$.

Next, let's have a closer look at the two equalities below.

$\mathbf{(\log_{15} 3)^2 + (\log_{15} 45)(\log_{15} 5) = 1}$, which is the equality we want to show.

$\mathbf{(\log_{15} 3)^2 + 2(\log_{15} 3)(\log_{15} 5) + (\log_{15} 5)^2 = 1}$, which is the expansion above.

Then, it looks we want to modify $(\log_{15} 45)(\log_{15} 5)$ to $2(\log_{15} 3)(\log_{15} 5) + (\log_{15} 5)^2$.

Quite often though in math, going backward seems to work better.

Thus, we may want to go from $2(\log_{15} 3)(\log_{15} 5) + (\log_{15} 5)^2$ to $(\log_{15} 45)(\log_{15} 5)$, and then, we can readily go back from $(\log_{15} 45)(\log_{15} 5)$ to $2(\log_{15} 3)(\log_{15} 5) + (\log_{15} 5)^2$.

To begin with, we can have $2(\log_{15} 3) = 2 \log_{15} 3 = \log_{15} 3^2 = \log_{15} 9$.

So we get $2(\log_{15} 3)(\log_{15} 5) + (\log_{15} 5)^2 = (\log_{15} 9)(\log_{15} 5) + (\log_{15} 5)^2$

$= (\log_{15} 9 + \log_{15} 5)(\log_{15} 5) = (\log_{15} 45)(\log_{15} 5)$.

That's because we can have $\log_{15} 9 + \log_{15} 5 = \log_{15} 9{\cdot}5 = \log_{15} 45$.

Thus, we get $2(\log_{15} 3)(\log_{15} 5) + (\log_{15} 5)^2 = (\log_{15} 45)(\log_{15} 5)$.

In short:

To begin with, $\dfrac{\log_{15} 45}{\log_5 15} = \log_{15} 45 \cdot \dfrac{1}{\log_5 15}$.

Meanwhile, $\dfrac{1}{\log_5 15} = \dfrac{1}{\frac{\log_{15} 15}{\log_{15} 5}} = \dfrac{1}{\frac{1}{\log_{15} 5}} = (\log_{15} 5)$.

So $\dfrac{\log_{15} 45}{\log_5 15} = \log_{15} 45 \cdot \dfrac{1}{\log_5 15} = (\log_{15} 45)(\log_{15} 5)$.

Next, $(\log_{15} 45)(\log_{15} 5) = (\log_{15} 5 + \log_{15} 9)(\log_{15} 5) =(\log_{15} 5 + 2 \log_{15} 3)(\log_{15} 5)$

$= (\log_{15} 5)^2 + 2(\log_{15} 3)(\log_{15} 5)$.

Thus, $(\log_{15} 3)^2 + \dfrac{\log_{15} 45}{\log_5 15}$

$= (\log_{15} 3)^2 + (\log_{15} 5)^2 + 2(\log_{15} 3)(\log_{15} 5) = (\log_{15} 3 + \log_{15} 5)^2 = (\log_{15} 15)^2 = 1$.

Suggestions or Solutions
To the Problem in the Example 2

Suppose that a, b, and $c > 0$, but that a, b, and $c \neq 1$.
Suppose also, that we have these:
p, q, and $r \neq 0$.
$pq + qr + rp = pqr$.
$\log_a t = p$, $\log_b t = q$, and $\log_c t = r$.

Then, put t in terms of a, b, and c.

To begin with, $\log_a t = p \Rightarrow \dfrac{1}{\log_t a} = p \Rightarrow \log_t a = \dfrac{1}{p}$.

$\log_b t = q \Rightarrow \dfrac{1}{\log_t b} = q \Rightarrow \log_t b = \dfrac{1}{q}$. $\log_c t = r \Rightarrow \dfrac{1}{\log_t c} = r \Rightarrow \log_t c = \dfrac{1}{r}$.

Next, $pq + qr + rp = pqr \Rightarrow \dfrac{1}{r} + \dfrac{1}{p} + \dfrac{1}{q} = 1$.

So we get $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1 \Rightarrow \log_t a + \log_t b + \log_t c = 1$.

Also, $\log_t a + \log_t b + \log_t c = \log_t abc$, too.

Therefore, $\log_t abc = 1 \Rightarrow abc = t^1 = t \Rightarrow t = abc$.

If not quite sure of the idea behind the processes above, follow the steps below.

Examining expressions above, we can notice that p, q, and r are all in the same situation, because p, q, and r are connected to each other via the expression, $pq + qr + rp = pqr$.

In fact, they all have the same condition, too, where each of them $\neq 0$.

Also, looking at equalities, $\log_a t = p$, $\log_b t = q$, and $\log_c t = r$, we can notice a and p are correlated via t, b and q are correlated via t, and so are c and r.

Besides, a, b, and c all have the same condition, too, where they are all > 0, and $\neq 1$.

Thus, we can see that a, b, and c are all in the same situation, too.
So where do we begin, and what do we work with?

Examining again, all the expressions given, we can notice that t is common in the three equalities, and all the other, a, b, etc. are in the three, too.

So putting t in terms of a, b, and c, we seem to want to begin with the three equalities as follows. $\log_a t = p$, $\log_b t = q$, and $\log_c t = r$.

However, all logs in the equalities have different bases. In math, consistency matters. Thus, we may want to let them have the same base.

Using a log identity, $\log_x y = \dfrac{1}{\log_y x}$, we can make each have a base t. Then, we get

$\log_a t = p \Rightarrow \dfrac{1}{\log_t a} = p \Rightarrow \log_t a = \dfrac{1}{p}$. $\log_b t = q \Rightarrow \dfrac{1}{\log_t b} = q \Rightarrow \log_t b = \dfrac{1}{q}$.
$\log_c t = r \Rightarrow \dfrac{1}{\log_t c} = r \Rightarrow \log_t c = \dfrac{1}{r}$. However, we are not given $\dfrac{1}{p}$, $\dfrac{1}{q}$, and $\dfrac{1}{r}$.

We can come up with those, though. Taking a closer look at the expression, $pq + qr + rp = pqr$, we can see the fractions there. How?

Examining each term in the expression closely, we can notice that the divisions of both sides by pqr can give us the ones we want.

We can do so, since $pqr \neq 0$, because p, q, and $r \neq 0$.

So doing such divisions to both of the sides, we get this:

$\boldsymbol{pq + qr + rp = pqr \Rightarrow \frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1}$.

Now, via the last expression above, we can put together all the three logs as follows.

$\boldsymbol{\log_t a = \frac{1}{p}}$, $\boldsymbol{\log_t b = \frac{1}{q}}$, and $\boldsymbol{\log_t c = \frac{1}{r}}$.

Then, we get $\boldsymbol{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \Rightarrow \log_t a + \log_t b + \log_t c = 1}$.

Besides, we have $\boldsymbol{\log_t a + \log_t b + \log_t c = \log_t abc}$, too.

So we get $\boldsymbol{\log_t abc = 1 \Rightarrow abc = t^1 = t}$. Therefore, $\boldsymbol{t = abc}$.

In short:

To begin with, $\boldsymbol{\log_a t = p \Rightarrow \frac{1}{\log_t a} = p \Rightarrow \log_t a = \frac{1}{p}}$.

$\boldsymbol{\log_b t = q \Rightarrow \frac{1}{\log_t b} = q \Rightarrow \log_t b = \frac{1}{q}}$. $\boldsymbol{\log_c t = r \Rightarrow \frac{1}{\log_t c} = r \Rightarrow \log_t c = \frac{1}{r}}$.

Next, $\boldsymbol{pq + qr + rp = pqr \Rightarrow \frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1}$.

So we get $\boldsymbol{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \Rightarrow \log_t a + \log_t b + \log_t c = 1}$.

Also, $\boldsymbol{\log_t a + \log_t b + \log_t c = \log_t abc}$.

Therefore, $\boldsymbol{\log_t abc = 1 \Rightarrow abc = t^1 = t \Rightarrow t = abc}$.

Suggestions or Solutions
To the Problem in the Example 3

Assuming $1 < c < d$, and $c^{\frac{1}{m}} = d^{\frac{1}{n}} = cd$, show this: $\sqrt{m^3 - m^2n - mn^2 + n^3} = \log_{dc} \frac{d}{c}$.

To begin with, $c^{\frac{1}{m}} = d^{\frac{1}{n}} = cd \Rightarrow \log c^{\frac{1}{m}} = \log d^{\frac{1}{n}} = \log cd \Rightarrow \frac{1}{m}\log c = \frac{1}{n}\log d = \log cd$.

So we get $\frac{1}{m}\log c = \log cd$, and $\frac{1}{n}\log d = \log cd$.

We know that cd is the base in the $\log_{dc} \frac{d}{c}$, so $cd \neq 1$, and thus, $\log cd \neq 0$.

Next, we get these:

$$\frac{1}{m}\log c = \log cd \Rightarrow \log c = m\log cd \Rightarrow m = \frac{\log c}{\log cd}.$$

$$\frac{1}{n}\log d = \log cd \Rightarrow \log d = n\log cd \Rightarrow n = \frac{\log d}{\log cd}.$$

$$m^3 - m^2n - mn^2 + n^3 = m^2(m - n) - n^2(m - n) = (m^2 - n^2)(m - n) = (m + n)(m - n)^2.$$

We have $m = \frac{\log c}{\log cd}$, and $n = \frac{\log d}{\log cd}$. Thus, we get these:

$$m + n = \frac{\log c}{\log cd} + \frac{\log d}{\log cd} = \frac{\log c + \log d}{\log cd} = \frac{\log cd}{\log cd} = 1.$$

$$m - n = \frac{\log c}{\log cd} - \frac{\log d}{\log cd} = \frac{\log c - \log d}{\log cd} = \frac{\log \frac{c}{d}}{\log cd} = \log_{cd} \frac{c}{d}.$$

So we get $m^3 - m^2n - mn^2 + n^3 = (m + n)(m - n)^2 = (m - n)^2$.

$1 < c < d \Rightarrow \log 1 < \log c < \log d \Rightarrow 0 < \log c < \log d \Rightarrow \log c - \log d < 0$.

$1 < c < d \Rightarrow 1 < cd$, since both c and $d > 1$. So $1 < cd \Rightarrow \log 1 < \log cd \Rightarrow 0 < \log cd$.

Thus, we get $\boldsymbol{m-n=\dfrac{\log c-\log d}{\log cd}<0}$.

So we get $\boldsymbol{\sqrt{m^3-m^2n-mn^2+n^3}=n-m=\dfrac{\log d-\log c}{\log cd}=\dfrac{\log\frac{d}{c}}{\log cd}=\log_{cd}\frac{d}{c}}$.

Therefore, $\boldsymbol{\sqrt{m^3-m^2n-mn^2+n^3}=\log_{dc}\frac{d}{c}}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Why do you do this example?

As many other examples in this book, this example, too, is for your practice on log algebra. That's simply because algebra matters. Following procedures and steps, or understanding processes when taking courses in math, particularly calculus, or other courses having much to do with math, you've got to do algebra.

Once understood ideas, formulas, identities, rules or laws in math, real caliber is up to the amount and quality of examples. Understanding is one thing, and doing it is another. Practice doesn't make things perfect, since nothing is perfect, but can make them better.

Now, what have we got in this example?

Finding $\boldsymbol{m}$ and $\boldsymbol{n}$, first, and then, putting them into the expression, $\boldsymbol{m^3-m^2n-mn^2+n^3}$, we should end up with the square of the right hand side, that is, we should get $\boldsymbol{(\log_{dc}\frac{d}{c})^2}$.

Finding $\boldsymbol{m}$ and $\boldsymbol{n}$, we want to solve $\boldsymbol{c^{\frac{1}{m}}=d^{\frac{1}{n}}=cd}$, which is a system of two equations, one of which is $\boldsymbol{c^{\frac{1}{m}}=cd}$, and the other is $\boldsymbol{d^{\frac{1}{n}}=cd}$.

However, $\boldsymbol{m}$ and $\boldsymbol{n}$ are in the positions where exponents are. What can we do then?

We can get them out of there taking logs. Usually, extracting an exponent from a power, we take a log of it (or apply a log to it). So applying logs to the expression, we get

$$\boldsymbol{c^{\frac{1}{m}}=d^{\frac{1}{n}}=cd\Rightarrow\log c^{\frac{1}{m}}=\log d^{\frac{1}{n}}=\log cd\Rightarrow\tfrac{1}{m}\log c=\tfrac{1}{n}\log d=\log cd}.$$

Thus, we get $\frac{1}{m}\log c = \log cd$, and $\frac{1}{n}\log d = \log cd$. So we get

$$\frac{1}{m}\log c = \log cd \Rightarrow \frac{\log c}{m} = \log cd \Rightarrow \log c = m\log cd \Rightarrow m = \frac{\log c}{\log cd}.$$

And we get this, too:

$\frac{1}{n}\log d = \log cd \Rightarrow \log d = n\log cd \Rightarrow n = \frac{\log d}{\log cd}$. What if **log *cd* = 0**, though?

We know that ***cd*** is the base in the $\log_{dc}\frac{d}{c}$, so $cd \neq 1$, and thus, $\log cd \neq 0$.

So ***m*** and ***n*** both are OK with the denominator **log *cd***.

Now, let's move on to the expression, $m^3 - m^2n - mn^2 + n^3$.

Before doing the substitutions however, we may want to examine the expression above.

That's because it seems that the substitutions will make the expression quite messy, so there probably be something we can do about the expression prior to the substitutions.

Closely looking at the expression, we can notice that it is quite symmetric, and can see that both ***m*** and ***n*** have the same situation. Therefore, we may want to try factorizing the expression. To begin with, we can put it the way below, too.

$m^3 - m^2n - mn^2 + n^3 = m^3 - m^2n + n^3 - n^2m$.

Factorizing the above, we get

$m^3 - m^2n + n^3 - n^2m = m^2(m - n) + n^2(n - m) = m^2(m - n) - n^2(m - n)$

$= (m^2 - n^2)(m - n) = (m + n)(m - n)(m - n) = (m + n)(m - n)^2$, which looks quite simpler now.

So next, we may want to find $m + n$ and $m - n$, first.

We have $m = \frac{\log c}{\log cd}$, and $n = \frac{\log d}{\log cd}$. Thus, we get these:

$$m + n = \frac{\log c}{\log cd} + \frac{\log d}{\log cd} = \frac{\log c + \log d}{\log cd} = \frac{\log cd}{\log cd} = 1.$$

$$m - n = \frac{\log c}{\log cd} - \frac{\log d}{\log cd} = \frac{\log c - \log d}{\log cd} = \frac{\log \frac{c}{d}}{\log cd} = \log_{cd} \tfrac{c}{d}.$$

So we get $m^3 – m^2n – mn^2 + n^3 = (m + n)(m – n)^2 = (m – n)^2$, since $m + n = 1$.

Now, note that $\sqrt{m^3 - m^2n - mn^2 + n^3} \geq 0$, which is obvious, since a square root of any real number is greater than or equal to 0 if no minus sign is in front of it.

And what's inside the square root sign is ≥ 0, of course, since the root itself has to be real. That is, $\sqrt{m^3 - m^2n - mn^2 + n^3}$ is real if $m^3 – m^2n – mn^2 + n^3 \geq 0$.

Working with any power in this form: $b^{\frac{1}{n}}$ where n is even, make sure that $b \geq 0$.

If n is even and $b \geq 0$, we can set $b^{\frac{1}{n}} = \sqrt[n]{b}$, where $\sqrt[n]{\ }$ is called a radical sign, and is more specifically, called an n^{th} root sign.

However, when we get to remove such a radical sign, we can make a mistake.

Removing such a radical sign when n is even, we want to make sure what's inside the radical sign is ≥ 0.

Suppose for instance, $y = \sqrt[4]{a^4}$.

Then, we can't just set $y = a$, since we don't know if $a \geq 0$. So we want to set $y = |a|$.

So suppose for instance, we have $y = \sqrt[4]{a^4}$, where $a < 0$.

Then, we need to set $y = -a$, because $y \geq 0$ and $a < 0$.

Now, we have $\sqrt{\boldsymbol{m^3 - m^2n - mn^2 + n^3}} = \sqrt{\boldsymbol{(m-n)^2}} \geq \mathbf{0}$.

So we want to check to see if $\boldsymbol{m - n \geq 0}$ before removing the radical sign (the square root sign) and the exponent, 2 from $\boldsymbol{(m - n)^2}$.

If $\boldsymbol{m - n \geq 0}$, we need to take $\boldsymbol{m - n}$.

Otherwise, we want to take $\boldsymbol{n - m}$ after the removal, since $\boldsymbol{m - n < 0}$.

That is:

If $\boldsymbol{m - n \geq 0}$, we get $\sqrt{\boldsymbol{(m-n)^2}} = \boldsymbol{m - n}$.

If $\boldsymbol{m - n < 0}$, we get $\sqrt{\boldsymbol{(m-n)^2}} = \boldsymbol{-(m-n) = n - m}$.

So let's give it a check now. To begin with, we have $\boldsymbol{1 < c < d}$.

Thus, $\boldsymbol{1 < c < d \Rightarrow \log 1 < \log c < \log d \Rightarrow 0 < \log c < \log d \Rightarrow \log c - \log d < 0}$.

On the other hand, we can get this, too: $\boldsymbol{1 < c < d \Rightarrow 1 < cd}$, since both $\boldsymbol{c}$ and $\boldsymbol{d > 1}$.

So we get $\boldsymbol{1 < cd \Rightarrow \log 1 < \log cd \Rightarrow 0 < \log cd}$.

Thus, we can see that $\boldsymbol{m - n = \dfrac{\log c - \log d}{\log cd} < 0}$, so we want to take ($\boldsymbol{n - m}$), which is $> \mathbf{0}$.

So we get $\boldsymbol{n - m = \dfrac{\log d - \log c}{\log cd} = \dfrac{\log \frac{d}{c}}{\log cd} = \log_{cd} \frac{d}{c}}$.

Thus, we get $\sqrt{\boldsymbol{m^3 - m^2n - mn^2 + n^3}} = \boldsymbol{n - m = \log_{cd} \frac{d}{c} = \log_{dc} \frac{d}{c}}$, since $\boldsymbol{cd = dc}$.

Therefore, $\sqrt{\boldsymbol{m^3 - m^2n - mn^2 + n^3}} = \boldsymbol{\log_{dc} \frac{d}{c}}$.

And we can put all the processes above in sum the way below.

To begin with, $c^{\frac{1}{m}} = d^{\frac{1}{n}} = cd \Rightarrow \log c^{\frac{1}{m}} = \log d^{\frac{1}{n}} = \log cd \Rightarrow \frac{1}{m}\log c = \frac{1}{n}\log d = \log cd$.

So we get $\frac{1}{m}\log c = \log cd$, and $\frac{1}{n}\log d = \log cd$.

We know that cd is the base in the $\log_{dc} \frac{d}{c}$, so $cd \neq 1$, and thus, $\log cd \neq 0$.

Next, we get thses:

$$\frac{1}{m}\log c = \log cd \Rightarrow \log c = m\log cd \Rightarrow m = \frac{\log c}{\log cd}.$$

$$\frac{1}{n}\log d = \log cd \Rightarrow \log d = n\log cd \Rightarrow n = \frac{\log d}{\log cd}.$$

$$m^3 - m^2n - mn^2 + n^3 = m^2(m - n) - n^2(m - n) = (m^2 - n^2)(m - n) = (m + n)(m - n)^2.$$

And we have $m = \dfrac{\log c}{\log cd}$, and $n = \dfrac{\log d}{\log cd}$. Thus, we get these:

$$m + n = \frac{\log c}{\log cd} + \frac{\log d}{\log cd} = \frac{\log c + \log d}{\log cd} = \frac{\log cd}{\log cd} = 1.$$

$$m - n = \frac{\log c}{\log cd} - \frac{\log d}{\log cd} = \frac{\log c - \log d}{\log cd} = \frac{\log \frac{c}{d}}{\log cd} = \log_{cd} \frac{c}{d}.$$

So we get $m^3 - m^2n - mn^2 + n^3 = (m + n)(m - n)^2 = (m - n)^2$.

$1 < c < d \Rightarrow \log 1 < \log c < \log d \Rightarrow 0 < \log c < \log d \Rightarrow \log c - \log d < 0$.

$1 < c < d \Rightarrow 1 < cd$, since both c and $d > 1$. So $1 < cd \Rightarrow \log 1 < \log cd \Rightarrow 0 < \log cd$.

Thus, we get $m - n = \dfrac{\log c - \log d}{\log cd} < 0$.

So we get $\sqrt{m^3 - m^2n - mn^2 + n^3} = n - m = \dfrac{\log d - \log c}{\log cd} = \dfrac{\log \frac{d}{c}}{\log cd} = \log_{cd} \frac{d}{c}$.

Therefore, $\sqrt{m^3 - m^2n - mn^2 + n^3} = \log_{dc} \frac{d}{c}$.