Examples 6 on Logarithms

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In this set of examples, too, we are going to do some more practice on algebra on logs so that you get more familiar with manipulation of logs. That's because algebra matters.

0. Assuming $\mathbf{0 < \boldsymbol{b} < 1 < \boldsymbol{a}}$, show that $\mathbf{\log_{\boldsymbol{a}} \boldsymbol{b} < 0}$.

1. Assuming that $\mathbf{1 < \boldsymbol{b} < \boldsymbol{a}}$, and $\boldsymbol{p} = (\log_a b)^2$, $\boldsymbol{q} = \log_a b^2$, and that $\boldsymbol{r} = \log_a (\log_a b)$, put $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ in an ascending order. Note that $\log_a (\log_a b) = \log_a \log_a b$, so the use of parentheses in this case is merely for clarity purposes.

2. Assuming $a^2 + b^2 = c^2$, show that $\log_{b+c} a + \log_{c-b} a = 2(\log_{b+c} a)(\log_{c-b} a)$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming $0 < b < 1 < a$, show that $\log_a b < 0$.

Since $\mathbf{0 < b < 1 < a}$, we get $\mathbf{b < 1 \Rightarrow \log_a b < \log_a 1 = 0 \Rightarrow \log_a b < 0}$.

We can put it this way, too:

$\mathbf{b < 1 \Rightarrow \log b < \log 1 = 0 \Rightarrow \log b < 0}$. $\mathbf{1 < a \Rightarrow \log 1 < \log a \Rightarrow 0 < \log a}$.

Therefore, $\dfrac{\log b}{\log a} < 0$, so we get $\log_a b < 0$, since $\log_a b = \dfrac{\log b}{\log a}$.

If not quite sure of the idea behind the processes above, follow the steps below.

From the relational expression given, we can get two relational expressions as follows.
$\mathbf{0 < b < 1}$, and $\mathbf{1 < a}$.

Using two expressions above, we should be able to draw the conclusion that $\mathbf{\log_a b < 0}$.
Of the two expressions, let's begin with the first.

The first one is $\mathbf{0 < b < 1}$, which means at the same time two cases as follows.
$\mathbf{0 < b}$ and $\mathbf{b < 1}$.

Then first, since $\mathbf{b > 0}$, we can take a log of $\mathbf{b}$.

Next, let's move on to the expression, $\mathbf{b < 1}$.

Suppose now, $\mathbf{1 < p}$.

Then, applying a $\mathbf{\log_p}$ to the expression $\mathbf{b < 1}$, the direction of the inequality sign remains the same, since the base $\mathbf{p > 1}$. If $\mathbf{0 < p < 1}$ however, the direction changes.

For instance, we have $\frac{1}{2} > \frac{1}{8}$, and $\mathbf{0.04 < 0.008}$.

Then, we get $\log_2 \frac{1}{2} > \log_2 \frac{1}{8}$, because $\log_2 \frac{1}{2} = \log_2 2^{-1} = -1$, and $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$.

So the direction of the inequality sign remains the same if the base > **1**.

And the same is true for any antilog positive, too.

For instance, $\log_2 8 > \log_2 4$, because $\log_2 8 = \log_2 2^3 = 3$, and $\log_2 4 = \log_2 2^2 = 2$.

However, we get $\log_{0.2} 0.04 < \log_{0.2} 0.008$.

That's because $\log_{0.2} 0.04 = \log_{0.2} 0.2^2 = 2$, and $\log_{0.2} 0.008 = \log_{0.2} (0.2)^3 = 3$.

So the direction of the inequality sign changes if **0** < the base < **1**.

Now, let's move on to the second expression, $1 < a$.

Then, we get $b < 1 \Rightarrow \log_a b < \log_a 1 \Rightarrow \log_a b < 0$.

Suppose however, that $0 < a < 1$, and $0 < b < 1$.

Then, we get $b < 1 \Rightarrow \log_a b > \log_a 1 \Rightarrow \log_a b > 0$.

In short:

Since $0 < b < 1 < a$, we get $b < 1 \Rightarrow \log_a b < \log_a 1 = 0 \Rightarrow \log_a b < 0$.

We can put it this way, too:

$b < 1 \Rightarrow \log b < \log 1 = 0 \Rightarrow \log b < 0$. $1 < a \Rightarrow \log 1 < \log a \Rightarrow 0 < \log a$.

Therefore, $\frac{\log b}{\log a} < 0$, so we get $\log_a b < 0$, since $\log_a b = \frac{\log b}{\log a}$.

Suggestions or Solutions
To the Problem in the Example 1

Assuming that $1 < b < a$, and $p = (\log_a b)^2$, $q = \log_a b^2$, and that $r = \log_a (\log_a b)$, put p, q, and r in an ascending order. Note that $\log_a (\log_a b) = \log_a \log_a b$, so the use of parentheses in this case is merely for clarity purposes.

We have $p = (\log_a b)^2$, $q = \log_a b^2 = 2\log_a b$, and $r = \log_a (\log_a b)$.

Besides, $1 < b < a \Rightarrow \log_a 1 < \log_a b < \log_a a \Rightarrow 0 < \log_a b < 1$.

Also, $1 < b < a \Rightarrow a > 1$.

So we now have $a > 1$ and $0 < \log_a b < 1$.

Thus, $r = \log_a (\log_a b) < 0 \Rightarrow r < 0$.

Next, $q - p = 2 \log_a b - (\log_a b)^2 = (\log_a b)(2 - \log_a b)$.

Meanwhile, $0 < \log_a b < 1 \Rightarrow -1 < -\log_a b < 0 \Rightarrow 1 < 2 - \log_a b < 2$.

Thus, $(\log_a b)(2 - \log_a b) > 0 \Rightarrow q - p > 0 \Rightarrow q > p$.

Therefore, $q > p > r$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, let's take a closer look at the expressions given.
Then, we can notice that p, q, and r are respectively put in terms of $\log_a b$.   How?

In $\log_a (\log_a b)$, the antilog is $\log_a b$, and we can get $\log_a b^2 = 2\log_a b$.

Thus, we may want to come up with something significant to $\log_a b$. How?

We should be able to take advantage of the condition given. What condition?

We are given a condition that $1 < b < a$, in which we should be able to find significant information on that log, which is $\log_a b$.

Then, out of the condition, we can come up with an expression with the log, $\log_a b$.

So let's first, take advantage of the condition, $1 < b < a$.

In the condition, we can see that $1 < a$.

Thus, if we apply a $\log_a$ to the condition, which is a relational expression, the directions of the inequality signs remain the same, so we get

$1 < b < a \Rightarrow \log_a 1 < \log_a b < \log_a a \Rightarrow 0 < \log_a b < 1$.

Let's next, see what we can do about p, q and r respectively.

Then, setting $x = \log_a b$, we get $p = x^2$, $q = 2x$, and $r = \log_a x$, in each of which $0 < x < 1$, since $x = \log_a b$, and $0 < \log_a b < 1$.

Let's look at r, first.

We know that the base $a > 1$. So we get these:

$\log_a x < 0$ if $0 < x < 1$.

$\log_a x \geq 0$ if $1 \leq x$.

Thus, we can see that $r = \log_a x < 0$, since the base $a > 1$ and $0 < x < 1$. How?

Refer to the problem in the Example 4.

Let's now, move on to $\boldsymbol{p = x^2}$, and $\boldsymbol{q = 2x}$, where $\boldsymbol{x = \log_a b}$.

Then, we can readily see that both $\boldsymbol{p}$ and $\boldsymbol{q > 0}$ when $\boldsymbol{0 < x < 1}$, so $\boldsymbol{r}$ is the smallest of all, since $\boldsymbol{r = \log_a x < 0}$ when $\boldsymbol{0 < x < 1}$.

Thus, we now, want to compare $\boldsymbol{p}$ and $\boldsymbol{q}$.

Comparing two objects, we can take either of the two ways as follows.
One is that we take the difference between the two, and then, check to see if it is positive.
The other is that we take the ratio between the two, and check to see if it is greater than 1.

And taking the difference, we get $\boldsymbol{q - p = 2x - x^2 = x(2 - x) \Rightarrow q - p = x(2 - x)}$.

We have $\boldsymbol{0 < x < 1}$, too.

Thus, we get $\boldsymbol{0 < x < 1 \Rightarrow 0 > -x > -1 \Rightarrow 2 > (2 - x) > -1 + 2 = 1 \Rightarrow 1 < (2 - x) < 2}$.

So we can see that both $\boldsymbol{x}$ and $\boldsymbol{(2 - x)}$ are positive at least.

Thus, $\boldsymbol{q - p = x(2 - x) > 0 \Rightarrow q - p > 0 \Rightarrow q > p}$.

Now putting threads together, we get $\boldsymbol{q > p > r}$.

Suggestions or Solutions
To the Problem in the Example 2

Suppose $a^2 + b^2 = c^2$. Then, show that $\log_{b+c} a + \log_{c-b} a = 2(\log_{b+c} a)(\log_{c-b} a)$.

$$\log_{c+b} a + \log_{c-b} a = \frac{1}{\log_a c+b} + \frac{1}{\log_a c-b} = \frac{\log_a c-b + \log_a c+b}{(\log_a c+b)(\log_a c-b)} = \frac{\log_a (c-b)(c+b)}{(\log_a c+b)(\log_a c-b)}$$

$$= \frac{\log_a c^2 - b^2}{(\log_a c+b)(\log_a c-b)} = \frac{\log_a a^2}{(\log_a c+b)(\log_a c-b)} = \frac{2}{(\log_a c+b)(\log_a c-b)}.$$ Meanwhile,

$$\frac{1}{\log_a c+b} \cdot \frac{1}{\log_a c-b} = (\log_{c+b} a)(\log_{c-b} a).$$ So $\log_{c+b} a + \log_{c-b} a = 2(\log_{c+b} a)(\log_{c-b} a)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Closely looking at the equality in logs, we can see that all the logs have the same antilog, which is a, and are to bases same or similar. In fact, the equality is made of two different logs, which are $\log_{b+c} a$ and $\log_{c-b} a$. The only difference between the two is in the base.

Usually, it is a good idea to let logs have the same base. It is particularly the case if logs are in an expression as an equation, equality, etc. Now, all the logs in the equality above have the same antilog. So it is not going to be very hard to make the logs have the same base. The same base is going to be a, of course. We have a tool for such a work.

The tool is a log identity, where $\log_x y = \frac{1}{\log_y x}$ for x and y both > 0 but $\neq 1$.

So beginning with the left hand side in the equality, we get this:

$$\log_{c+b} a + \log_{c-b} a = \frac{1}{\log_a c+b} + \frac{1}{\log_a c-b} = \frac{\log_a c-b + \log_a c+b}{(\log_a c+b)(\log_a c-b)} = \frac{\log_a (c-b)(c+b)}{(\log_a c+b)(\log_a c-b)}$$

$$= \frac{\log_a c^2 - b^2}{(\log_a c+b)(\log_a c-b)} = \frac{\log_a a^2}{(\log_a c+b)(\log_a c-b)} = \frac{2}{(\log_a c+b)(\log_a c-b)}.$$

Meanwhile, we have $\frac{1}{\log_a c+b} \cdot \frac{1}{\log_a c-b} = (\log_{c+b} a)(\log_{c-b} a)$.

Thus, we get $\log_{c+b} a + \log_{c-b} a = 2(\log_{c+b} a)(\log_{c-b} a)$, which is the equality we had to show.

In addition, considering the structure of the expression $a^2 + b^2 = c^2$ given as a condition in this problem, we can notice that the equality below can hold, too.

$\log_{c+a} b + \log_{c-a} b = 2(\log_{c+a} b)(\log_{c-a} b)$.

That's because even if a and b exchange their positions, the expression $a^2 + b^2 = c^2$ remains the same. Not quite sure?

We have $a^2 + b^2 = c^2 \Rightarrow \log_{c+b} a + \log_{c-b} a = 2(\log_{c+b} a)(\log_{c-b} a)$.

Now, if a and b exchange their positions, we get

$a^2 + b^2 = c^2 \Rightarrow \log_{c+a} b + \log_{c-a} b = 2(\log_{c+a} b)(\log_{c-a} b)$. Still not quite sure?

First, the equality $\log_{c+b} a + \log_{c-b} a = 2(\log_{c+b} a)(\log_{c-b} a)$ can hold if $a^2 + b^2 = c^2$.

Next, having a and b exchange their positions, we get

$\log_{c+a} b + \log_{c-a} b = 2(\log_{c+a} b)(\log_{c-a} b)$ from $\log_{c+b} a + \log_{c-b} a = 2(\log_{c+b} a)(\log_{c-b} a)$, but still get $a^2 + b^2 = c^2$ from $a^2 + b^2 = c^2$.

That is, $a^2 + b^2 = c^2$ remains the same even if a and b exchange their positions.

So another equality $\mathbf{log}_{c+a}\, b + \mathbf{log}_{c-a}\, b = 2(\mathbf{log}_{c+a}\, b)(\mathbf{log}_{c-a}\, b)$ can hold, too, if $a^2 + b^2 = c^2$.

By the way, the equality $a^2 + b^2 = c^2$ can be called the right triangle identity, often called Pythagorean theorem, too, where the square of the hypotenuse equals the sum of the square of the base and that of the height. So in the equality $a^2 + b^2 = c^2$, c can be taken as a hypotenuse, and b and c are line segments perpendicular to each other.

There is another way we can verify this: $\mathbf{log}_{b+c}\, a + \mathbf{log}_{c-b}\, a = 2(\mathbf{log}_{b+c}\, a)(\mathbf{log}_{c-b}\, a)$.

We put in a fraction every log in the equality above, and then, compare both sides of the equality. So beginning with the left hand side, $\mathbf{log}_{b+c}\, a + \mathbf{log}_{c-b}\, a$, and setting

$\log_{b+c} a = \dfrac{\log a}{\log(b+c)}$, and $\log_{c-b} a = \dfrac{\log a}{\log(c-b)}$, we get this:

$$\log_{b+c} a + \log_{c-b} a = \log a\left(\frac{1}{\log(c+b)} + \frac{1}{\log(c-b)}\right) = \log a \cdot \frac{\log(c-b)+\log(c+b)}{\log(c+b)\log(c-b)}.$$

So we get $\log_{b+c} a + \log_{c-b} a = \log a \cdot \dfrac{\log(c-b)+\log(c+b)}{\log(c+b)\log(c-b)}$.

Meanwhile, we have this:

$$(\log a)\{\log (c-b) + \log (c+b)\} = (\log a)\{\log (c-b)(c+b)\} = (\log a)\{\log (c^2 - b^2)\}$$

$$= (\log a)(\log a^2) = (\log a)2(\log a) = 2(\log a)^2.$$

Thus, we get $\log_{b+c} a + \log_{c-b} a = \dfrac{2(\log a)^2}{\log(c+b)\log(c-b)}$.

Next, moving on to the right hand side, we get

$$\mathbf{2(\log_{b+c} a)(\log_{c-b} a) = \frac{2\log a}{\log(b+c)} \cdot \frac{\log a}{\log(c-b)} = \frac{2(\log a)^2}{\log(c+b)\log(c-b)}}$$, which is the same as the left hand side.

Let's this time, see if we can verify this: $\mathbf{\log_{c+a} b + \log_{c-a} b = 2(\log_{c+a} b)(\log_{c-a} b)}$.

Beginning with the left hand side, $\mathbf{\log_{c+a} b + \log_{c-a} b}$, and setting

$\mathbf{\log_{c+a} b = \frac{\log b}{\log(c+a)}}$, and $\mathbf{\log_{c-a} b = \frac{\log b}{\log(c-a)}}$, we get this:

$$\mathbf{\log_{c+a} b + \log_{c-a} b = \log b\left(\frac{1}{\log(c+a)} + \frac{1}{\log(c-a)}\right) = \log b \cdot \frac{\log(c-a)+\log(c+a)}{\log(c+a)\log(c-a)}}.$$

So we get $\mathbf{\log_{c+a} b + \log_{c-a} b = \log b \cdot \frac{\log(c-a)+\log(c+a)}{\log(c+a)\log(c-a)}}$.

Meanwhile, we have

$\mathbf{(\log b)\{\log (c - a) + \log (c + a)\} = (\log b)\{\log (c - a)(c + a)\} = (\log b)\{\log (c^2 - a^2)\}}$
$\mathbf{= (\log b)(\log b^2) = (\log b)2(\log b) = 2(\log b)^2}$.

Thus, we get $\mathbf{\log_{c+a} b + \log_{c-a} b = \frac{2(\log b)^2}{\log(c+a)\log(c-a)}}$.

Next, moving on to the right hand side, we get

$$\mathbf{2(\log_{c+a} b)(\log_{c-a} b) = \frac{2\log b}{\log(c+a)} \cdot \frac{\log b}{\log(c-a)} = \frac{2(\log b)^2}{\log(c+a)\log(c-a)}}$$, which is the same as the left hand side.

In short:

$$\log_{c+b} a + \log_{c-b} a = \frac{1}{\log_a c+b} + \frac{1}{\log_a c-b} = \frac{\log_a c-b+\log_a c+b}{(\log_a c+b)(\log_a c-b)} = \frac{\log_a (c-b)(c+b)}{(\log_a c+b)(\log_a c-b)}$$

$$= \frac{\log_a c^2 - b^2}{(\log_a c+b)(\log_a c-b)} = \frac{\log_a a^2}{(\log_a c+b)(\log_a c-b)} = \frac{2}{(\log_a c+b)(\log_a c-b)}.$$ Meanwhile,

$\frac{1}{\log_a c+b} \cdot \frac{1}{\log_a c-b} = (\log_{c+b} a)(\log_{c-b} a)$. So $\log_{c+b} a + \log_{c-b} a = 2(\log_{c+b} a)(\log_{c-b} a)$.

The other method:

On the left hand side of the equality, setting $\log_{b+c} a = \frac{\log a}{\log(b+c)}$, and $\log_{c-b} a = \frac{\log a}{\log(c-b)}$,

we get $\log_{b+c} a + \log_{c-b} a = \log a(\frac{1}{\log(c+b)} + \frac{1}{\log(c-b)}) = \log a \cdot \frac{\log(c-b)+\log(c+b)}{\log(c+b)\log(c-b)}$.

Meanwhile, $(\log a)\{\log (c - b) + \log (c + b)\} = (\log a)\{\log (c - b)(c + b)\}$

$= (\log a)\{\log (c^2 - b^2)\} = (\log a)(\log a^2) = (\log a)2(\log a) = 2(\log a)^2$.

Thus, we get $\log_{b+c} a + \log_{c-b} a = \frac{2(\log a)^2}{\log(c+b)\log(c-b)}$.

Next, on the right hand side, we get

$2(\log_{b+c} a)(\log_{c-b} a) = \frac{2\log a}{\log(b+c)} \cdot \frac{\log a}{\log(c-b)} = \frac{2(\log a)^2}{\log(c+b)\log(c-b)}$, which equals the left hand side.

Note that we have $2(\log_{b+c} a)(\log_{b+c} a) = 2\log_{b+c} a \log_{b+c} a$, so the parentheses in this case are for clarity purposes only.