Examples 7 on Logarithms

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In this set of examples, too, we are going to do some more practice on algebra on logs so that you get more familiar with manipulation of logs. That's because algebra matters.

0. Assuming $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{k}$ all are positive but not equal to 1, show that:
$\mathbf{2\log_{\mathit{b}} \mathit{k} = \log_{\mathit{a}} \mathit{k} + \log_{\mathit{c}} \mathit{k} \Leftrightarrow \mathit{c}^2 = (\mathit{ac})^{\mathit{t}}}$ where $\mathbf{\mathit{t} = \log_{\mathit{a}} \mathit{b}}$.

1. Solve simultaneous equations as follows. $\mathbf{\log_2 \mathit{x} + \log_3 \mathit{y} = 4}$, & $\mathbf{(\log_3 \mathit{x})(\log_2 \mathit{y}) = 3}$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming a, b, c, and k all are positive but not equal to 1, show that:
$2\log_b k = \log_a k + \log_c k \Leftrightarrow c^2 = (ac)^t$ where $t = \log_a b$.

Proof for: $2\log_b k = \log_a k + \log_c k \Rightarrow c^2 = (ac)^t$ where $t = \log_a b$.

$$2\log_b k = \frac{2\log k}{\log b}.$$

$$\log_a k + \log_c k = \frac{\log k}{\log a} + \frac{\log k}{\log c} = \log k \frac{(\log c + \log a)}{(\log a)(\log c)} = \frac{(\log k)(\log ac)}{(\log a)(\log c)}.$$

Then, $2\log_b k = \log_a k + \log_c k \Rightarrow \dfrac{2\log k}{\log b} = \dfrac{(\log k)(\log ac)}{(\log a)(\log c)} \Rightarrow \dfrac{2}{\log b} = \dfrac{\log ac}{(\log a)(\log c)}.$

$$\Rightarrow 2\log c = \frac{(\log ac)(\log b)}{(\log a)} = \log ac \cdot \frac{\log b}{\log a} = (\log ac)(\log_a b).$$

$\Rightarrow 2\log c = (\log ac)(\log_a b) \Rightarrow \log c^2 = (\log_a b)(\log ac).$

Setting $t = \log_a b$, we get $\log c^2 = t \log ac = \log (ac)^t$.

Therefore, $c^2 = (ac)^t$ where $t = \log_a b$.

Proof for: $c^2 = (ac)^t$ where $t = \log_a b \Rightarrow 2\log_b k = \log_a k + \log_c k$.

$c^2 = (ac)^t \Rightarrow \log c^2 = \log (ac)^t = t \log ac = (\log_a b)(\log ac) = \dfrac{\log b}{\log a} \cdot \log ac$

$\Rightarrow \log c^2 = \dfrac{\log b}{\log a} \cdot \log ac \quad \Rightarrow 2 \log c = \dfrac{\log b}{\log a} \cdot \log ac$

$\Rightarrow \dfrac{2}{\log b} = \dfrac{\log ac}{(\log a)(\log c)} \Rightarrow \dfrac{2 \log k}{\log b} = \dfrac{(\log k)(\log ac)}{(\log a)(\log c)}$.

Then, we get $2 \log_b k = \dfrac{(\log k)(\log c + \log a)}{(\log a)(\log c)} = \log k \left(\dfrac{\log c}{(\log a)(\log c)} + \dfrac{\log a}{(\log a)(\log c)} \right)$

$= \log k \left(\dfrac{1}{\log a} + \dfrac{1}{\log c} \right) = \dfrac{\log k}{\log a} + \dfrac{\log k}{\log c} = \log_a k + \log_c k$.

Therefore, $2\log_b k = \log_a k + \log_c k$.

So putting threads together, we have

$2\log_b k = \log_a k + \log_c k \Rightarrow c^2 = (ac)^t$ where $t = \log_a b$.

$c^2 = (ac)^t$ where $t = \log_a b \Rightarrow 2\log_b k = \log_a k + \log_c k$.

Therefore, we get $2\log_b k = \log_a k + \log_c k \Leftrightarrow c^2 = (ac)^t$ where $t = \log_a b$.

If not quite sure of the idea behind the processes above, follow the steps below.

This is just another example for practice on log algebra. Thus, doing this one, we get more used to such algebra. In this example, we can notice that if an equality is made of logs and all the logs have the same antilog, all the same antilogs can get canceled.

Besides, a log is an exponent, and thus, very naturally, can also, be an exponent in a power, of course. For instance, we can have $5^{\log 3}$.

Now, examining all expressions given in this problem, we can notice that the equality on the left hand side of $\Leftrightarrow$ has $\boldsymbol{k}$s, but the one on the right hand side has none of those.

Thus, doing log algebra in this case, we will probably see all $\boldsymbol{k}$s getting canceled.

Besides, we will notice that doing log algebra, we put a log in fractional form very often.

Let's now, begin with showing $\mathbf{2log}_{\boldsymbol{b}}\ \boldsymbol{k} = \mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{k} + \mathbf{log}_{\boldsymbol{c}}\ \boldsymbol{k} \Rightarrow \boldsymbol{c}^{2} = (\boldsymbol{ac})^{\boldsymbol{t}}$ where $\boldsymbol{t} = \mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{b}$, and see how log algebra goes.

In other words, we break ($\mathbf{2log}_{\boldsymbol{b}}\ \boldsymbol{k} = \mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{k} + \mathbf{log}_{\boldsymbol{c}}\ \boldsymbol{k}$) into pieces, and then, put the pieces together to form ($\boldsymbol{c}^{2} = (\boldsymbol{ac})^{\boldsymbol{t}}$ where $\boldsymbol{t} = \mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{b}$).

To begin with, we have a log identity, $\log_x y = \dfrac{\log y}{\log x}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ both $> \mathbf{0}$ but $\neq \mathbf{1}$.

So we can get

$$2\log_b k = 2\cdot\frac{\log k}{\log b} = \frac{2\log k}{\log b}, \text{ and } \log_a k + \log_c k = \frac{\log k}{\log a} + \frac{\log k}{\log c} = \log k\left(\frac{1}{\log a} + \frac{1}{\log c}\right).$$

Meanwhile, $\dfrac{1}{\log a} + \dfrac{1}{\log c} = \dfrac{(\log c + \log a)}{(\log a)(\log c)} = \dfrac{\log ac}{(\log a)(\log c)}$.

Thus, we get $\log_a k + \log_c k = \dfrac{(\log k)(\log ac)}{(\log a)(\log c)}$. And we have $2\log_b k = \dfrac{2\log k}{\log b}$, too.

So we get $2\log_b k = \log_a k + \log_c k \Rightarrow \dfrac{2\log k}{\log b} = \dfrac{(\log k)(\log ac)}{(\log a)(\log c)} \Rightarrow \dfrac{2}{\log b} = \dfrac{\log ac}{(\log a)(\log c)}$.

At this point, we may want to come up with $\mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{b}$, which is in the expression $\boldsymbol{t} = \mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{b}$, which is the exponent on the right hand side in the equality $\boldsymbol{c}^{2} = (\boldsymbol{ac})^{\boldsymbol{t}}$.

Now, we are showing this: $\mathbf{2log}_{\boldsymbol{b}}\ \boldsymbol{k} = \mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{k} + \mathbf{log}_{\boldsymbol{c}}\ \boldsymbol{k} \Rightarrow \boldsymbol{c}^{2} = (\boldsymbol{ac})^{\boldsymbol{t}}$ where $\boldsymbol{t} = \mathbf{log}_{\boldsymbol{a}}\ \boldsymbol{b}$.

We now have $\mathbf{2\log_b k = \log_a k + \log_c k \Rightarrow \frac{2}{\log b} = \frac{\log ac}{(\log a)(\log c)}}$.

So what we want to show now is this: $\mathbf{\frac{2}{\log b} = \frac{\log ac}{(\log a)(\log c)} \Rightarrow c^2 = (ac)^t}$.

In $\mathbf{c^2 = (ac)^t}$, we have $\mathbf{c^2}$ on the left hand side, and we have 2 in the numerator of $\mathbf{\frac{2}{\log b}}$.

And we know $\mathbf{2\log c = \log c^2}$.

So multiplying by ($\mathbf{\log c}$)($\mathbf{\log b}$) each side in the equality $\mathbf{\frac{2}{\log b} = \frac{\log ac}{(\log a)(\log c)}}$, we get

$$\mathbf{2\log c = \frac{(\log ac)(\log b)}{(\log a)} = \log ac \cdot \frac{\log b}{\log a} = (\log ac)(\log_a b)}.$$

So we get $\mathbf{2\log c = (\log ac)(\log_a b) \Rightarrow \log c^2 = (\log_a b)(\log ac)}$.

And we know $\mathbf{x \log_b A = \log_b A^x}$.

So setting $\mathbf{t = \log_a b}$, we get $\mathbf{\log c^2 = t \log ac = \log (ac)^t \Rightarrow \log c^2 = \log (ac)^t}$.

Therefore, we get $\mathbf{c^2 = (ac)^t}$ where $\mathbf{t = \log_a b}$.

So we can now say this: $\mathbf{2\log_b k = \log_a k + \log_c k \Rightarrow c^2 = (ac)^t}$ where $\mathbf{t = \log_a b}$.

And next, we want to show $\mathbf{c^2 = (ac)^t}$ where $\mathbf{t = \log_a b \Rightarrow 2\log_b k = \log_a k + \log_c k}$.

To begin with, taking logs of both sides in $\mathbf{c^2 = (ac)^t}$, we get

$$\mathbf{c^2 = (ac)^t \Rightarrow \log c^2 = \log (ac)^t = t \log ac = (\log_a b)(\log ac) = \frac{\log b}{\log a} \cdot \log ac}$$

$$\mathbf{\Rightarrow \log c^2 = \frac{\log b}{\log a} \cdot \log ac \quad \Rightarrow 2\log c = \frac{\log b}{\log a} \cdot \log ac}.$$

Dividing both sides by **(log *b*)(log *c*)** respectively, we get $\frac{\mathbf{2}}{\mathbf{\log} \boldsymbol{b}} = \frac{\mathbf{\log} \boldsymbol{ac}}{(\mathbf{\log} \boldsymbol{a})(\mathbf{\log} \boldsymbol{c})}$.

Multiplying both sides by **log *k*** respectively, we get $\frac{\mathbf{2\log} \boldsymbol{k}}{\mathbf{\log} \boldsymbol{b}} = \frac{(\mathbf{\log} \boldsymbol{k})(\mathbf{\log} \boldsymbol{ac})}{(\mathbf{\log} \boldsymbol{a})(\mathbf{\log} \boldsymbol{c})}$.

And we know $\frac{\mathbf{\log} \boldsymbol{k}}{\mathbf{\log} \boldsymbol{b}} = \mathbf{\log}_{b} \boldsymbol{k}$, and $\mathbf{\log} \boldsymbol{ac} = \mathbf{\log} \boldsymbol{a} + \mathbf{\log} \boldsymbol{c}$.

So we get $\mathbf{2\log}_{b} \boldsymbol{k} = \frac{(\mathbf{\log} \boldsymbol{k})(\mathbf{\log} \boldsymbol{a} + \mathbf{\log} \boldsymbol{c})}{(\mathbf{\log} \boldsymbol{a})(\mathbf{\log} \boldsymbol{c})} = \mathbf{\log} \boldsymbol{k}\left(\frac{\mathbf{\log} \boldsymbol{a}}{(\mathbf{\log} \boldsymbol{a})(\mathbf{\log} \boldsymbol{c})} + \frac{\mathbf{\log} \boldsymbol{c}}{(\mathbf{\log} \boldsymbol{a})(\mathbf{\log} \boldsymbol{c})}\right)$

$$= \mathbf{\log} \boldsymbol{k}\left(\frac{\mathbf{1}}{\mathbf{\log} \boldsymbol{c}} + \frac{\mathbf{1}}{\mathbf{\log} \boldsymbol{a}}\right) = \frac{\mathbf{\log} \boldsymbol{k}}{\mathbf{\log} \boldsymbol{a}} + \frac{\mathbf{\log} \boldsymbol{k}}{\mathbf{\log} \boldsymbol{c}} = \mathbf{\log}_{a} \boldsymbol{k} + \mathbf{\log}_{c} \boldsymbol{k}.$$

Thus, we get $\mathbf{2\log}_{b} \boldsymbol{k} = \mathbf{\log}_{a} \boldsymbol{k} + \mathbf{\log}_{c} \boldsymbol{k}$.

So we now have shown both below.

$\mathbf{2\log}_{b} \boldsymbol{k} = \mathbf{\log}_{a} \boldsymbol{k} + \mathbf{\log}_{c} \boldsymbol{k} \Rightarrow \boldsymbol{c}^{\mathbf{2}} = (\boldsymbol{ac})^{\boldsymbol{t}}$ where $\boldsymbol{t} = \mathbf{\log}_{a} \boldsymbol{b}$.

$\boldsymbol{c}^{\mathbf{2}} = (\boldsymbol{ac})^{\boldsymbol{t}}$ where $\boldsymbol{t} = \mathbf{\log}_{a} \boldsymbol{b} \Rightarrow \mathbf{2\log}_{b} \boldsymbol{k} = \mathbf{\log}_{a} \boldsymbol{k} + \mathbf{\log}_{c} \boldsymbol{k}$.

Therefore, we get $\mathbf{2\log}_{b} \boldsymbol{k} = \mathbf{\log}_{a} \boldsymbol{k} + \mathbf{\log}_{c} \boldsymbol{k} \Leftrightarrow \boldsymbol{c}^{\mathbf{2}} = (\boldsymbol{ac})^{\boldsymbol{t}}$ where $\boldsymbol{t} = \mathbf{\log}_{a} \boldsymbol{b}$.

Suggestions or Solutions
To the Problem in the Example 1

Solve simultaneous equations as follows: $\log_2 x + \log_3 y = 4$, and $(\log_3 x)(\log_2 y) = 3$.

$(\log_3 x)(\log_2 y) = 3 \Rightarrow (\log_2 x)(\log_3 y) = 3$.

Thus, we get $\log_2 x + \log_3 y = 4$, and $(\log_2 x)(\log_3 y) = 3$.

Setting $u = \log_2 x$, and $v = \log_3 y$, we get a system where $u + v = 4$ and $uv = 3$.

So u and v can be taken as the roots of an equation $x^2 - 4x + 3 = 0$.

$x^2 - 4x + 3 = (x - 3)(x - 1) = 0 \Rightarrow x = 3$ or 1.

Thus, we get $(u, v) = (3, 1)$ or $(1, 3)$.

So we get these:

$\log_2 x = 3 \Rightarrow x = 2^3 = 8$, and $\log_3 y = 1 \Rightarrow y = 3$.

$\log_2 x = 1 \Rightarrow x = 2$, and $\log_3 y = 3 \Rightarrow y = 3^3 = 27$.

Therefore, $(x, y) = (8, 3)$ or $(2, 27)$.

If not quite sure of the idea behind the processes above, follow the steps below.

All the tools in math are efficient. Of all the tools in math, substitutions are probably the most efficient.

In the system of equations given above, each unknown is an antilog in a log.

That is, unknowns are inside logs.   How then do we get access to the unknowns?

Getting the access, we don't need to isolate the unknowns. In fact, we may not want to try getting direct access to those. We can get indirect access, then get the values of the unknowns. That is to say that we may want to consult substitutions.   How though?

To begin with, we take for an unknown, $\log_2 x$ rather than x itself.
Next, we do the same to the others, too.

Then, we solve the system, and then, get the values of x and y.
So for instance, we can set $u = \log_2 x$, and then, move on to the algebra.

That's just one thing, though, in this case.
Another is that we have four different kinds. That is, we have for different logs.

Doing such substitutions, we get to have not two unknowns only, but actually have four, which are $\log_2 x$, $\log_3 y$, $\log_3 x$, and $\log_2 y$.

Well then we force them to be two only.   How though?

We can use another tool, which is as follows. $(\log_s P)(\log_t Q) = (\log_t P)(\log_s Q)$.

So we can swap the bases when two logs are multiplied by each other.

Let's first have a look at though, how we can get such a tool.

We have a log identity where $\log_x y = \dfrac{\log_c y}{\log_c x}$.

Thus, we get $\log_s P = \dfrac{\log_t P}{\log_t s}$, and $\log_t Q = \dfrac{\log_s Q}{\log_s t}$.

Meanwhile, $\log_s t = \dfrac{1}{\log_t s} \Rightarrow \log_t s = \dfrac{1}{\log_s t}$.

Thus, we get $\log_t Q = \dfrac{\log_s Q}{\log_s t} = (\log_s Q)(\log_t s)$. And we have $\log_s P = \dfrac{\log_t P}{\log_t s}$, too.

So we get $(\log_s P)(\log_t Q) = \dfrac{\log_t P}{\log_t s}(\log_s Q)(\log_t s) = (\log_t P)(\log_s Q)$.

Therefore, we get $(\log_3 x)(\log_2 y) = 3 \Rightarrow (\log_2 x)(\log_3 y) = 3$.

That is to say that we can swap the bases, which we can do of course, when two logs are multiplied by each other.

Now, we have $\log_2 x + \log_3 y = 4$, and $(\log_2 x)(\log_3 y) = 3$.

So setting $u = \log_2 x$, and $v = \log_3 y$, we get a system of $u + v = 4$ and $uv = 3$.

Solving for u the first of the two equations, we get $u = 4 - v$.

Next, putting $4 - v$ into u in $uv = 3$, we get an equation for v only, which is quadratic.

Then, we get to solve for v the quadratic equation.

And we can put the same the way below, too, of course.

We can take u and v for the solutions to a quadratic equation, $(x - u)(x - v) = 0$.

Then, expanding the equation above, we get $(x - u)(x - v) = x^2 - (u + v)x + uv = 0$.

And we know $u + v = 4$ and $uv = 3$.

So u and v are the roots of a quadratic equation, $x^2 - 4x + 3 = 0$.

Therefore, we should get the same equation via the following procedure, too.

$\boldsymbol{u = 4 - v \Rightarrow uv = 3 \Rightarrow (4 - v)v = 3 \Rightarrow 4v - v^2 = 3 \Rightarrow v^2 - 4v + 3 = 0}$.

What then about $\boldsymbol{u}$?

The same is true for $\boldsymbol{u}$, too.

Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are in the same situation in the system, we will get $\boldsymbol{u^2 - 4u + 3 = 0}$.

Thus, one of the two roots is the value of $\boldsymbol{u}$, and the other is the value of $\boldsymbol{v}$.

What do we mean by the same situation, though?

Even if $\boldsymbol{u}$ and $\boldsymbol{v}$ exchange their positions in the system where $\boldsymbol{u + v = 4}$ and $\boldsymbol{uv = 3}$, the system remains the same. So let's now, get the solution to $\boldsymbol{x^2 - 4x + 3 = 0}$. Then, we get

$\boldsymbol{x^2 - 4x + 3 = (x - 3)(x - 1) = 0 \Rightarrow x = 3}$ or $\boldsymbol{1}$.

Now, we know $\boldsymbol{x}$ is actually acting for $\boldsymbol{u}$ or $\boldsymbol{v}$.

Thus, we have two cases as follows. $\boldsymbol{u = 3}$ and $\boldsymbol{v = 1}$, or $\boldsymbol{u = 1}$ and $\boldsymbol{v = 3}$.

In sum, we can set $\boldsymbol{(u, v) = (3, 1)}$ or $\boldsymbol{(1, 3)}$.

We have $\boldsymbol{u = \log_2 x}$, and $\boldsymbol{v = \log_3 y}$.

Thus, we get these:

$\boldsymbol{\log_2 x = 3 \Rightarrow x = 2^3 = 8}$, and $\boldsymbol{\log_3 y = 1 \Rightarrow y = 3^1 = 3}$.

$\boldsymbol{\log_2 x = 1 \Rightarrow x = 2^1 = 2}$, and $\boldsymbol{\log_3 y = 3 \Rightarrow y = 3^3 = 27}$.

Therefore, $\boldsymbol{(x, y) = (8, 3)}$ or $\boldsymbol{(2, 27)}$.