Examples 8 on Logarithms

Click this to start.

Examples 8 on Logarithms

0. Assuming $\boldsymbol{n}$ is integer, and $\mathbf{1.25}^{n}$ has 8 digits above the decimal point, find $\boldsymbol{n}$. Use **log 2 = 0.3010**.

1. Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are positive integers, and that $\boldsymbol{a}^{\mathbf{100}}$ has 120 digits. Then:

1.0. Find the highest place value in $\boldsymbol{a}^{\mathbf{-1}}$.

1.1. Find the value of $\boldsymbol{b}$ that makes $\boldsymbol{a}^{b}$ a 10-digit integer.

Suggestions or Solutions
To the Problem in the Example 0

Assuming n is integer, and 1.25^n has 8 digits above the decimal point, find n.
Use $\log 2 = 0.3010$.

$\log 1.25^n = n \log 1.25 = 7 + m$, where $0 \le m < 1$.

$$\log 1.25 = \log \tfrac{125}{100} = \log \tfrac{1000}{800} = \log \tfrac{10}{8} = 1 - 3\log 2 = 1 - 3 \cdot 0.3010 = 1 - 0.9030 = 0.097.$$

Thus, $n \log 1.25 = 0.097n = 7 + m$.

Besides, we have $0 \le m < 1$, too.

Thus, $7 \le 7 + m < 8 \Rightarrow 7 \le 0.097n < 8 \Rightarrow \frac{7}{0.097} \le n < \frac{8}{0.097} \Rightarrow 72 + \frac{16}{97} \le n < 82 + \frac{46}{97}$.

Therefore, $n = 73, 74, 75, \ldots$, and 82.

If not quite sure of the idea behind the processes above, follow the steps below.

What does the number of digits in a number have to do with?

It has much to do with the highest place value in the number. How then can we get the place value?

We can get the information on it by taking a common log of the number.

Assuming the number is A, and taking a common log of it, we can set

$\log A = c + m$, where c is an integer, and $0 \le m < 1$.

Then, c is often called the characteristic, m is called the mantissa, and the highest place value in A is $\mathbf{10}^c$.

So if we have $A \geq 1$, we get $c \geq 0$, and the number of digits above the decimal point in A is $c + 1$.

If however, we have $0 < A < 1$, we get $c < 0$, so the first nonzero appears at the $|c|^{\text{th}}$ digit below the decimal point in A. For instance, if $c = -3$, the first nonzero appears at the 3rd digit below the decimal point.

Let's now, begin with taking a common log of the number given, which is $\mathbf{1.25}^n$.

Since the number, $\mathbf{1.25}^n$ has 8 digits above the point, the characteristic $c = 7$.

Thus, we can set $\log 1.25^n = n \log 1.25 = 7 + m$, where $0 \leq m < 1$.

So next, we want to get the value of **log 1.25**. So can we get the value from a log table?

No, we are not allowed to use a log table doing this problem. What else then can we do?

We are given the value of **log 2**, which is 0.3010, so we should be able to put **log 1.25** in terms of **log 2**. So we've got to do some log algebra, and doing it, we can get

$$1.25 = \tfrac{125}{100} = \tfrac{5}{4} = \tfrac{10}{8} \Rightarrow \log 1.25 = \log \tfrac{10}{8} = 1 - 3\log 2 = 1 - 3 \cdot 0.3010 = 1 - 0.9030 = 0.097.$$

Thus, we get $n \log 1.25 = 7 + m \Rightarrow 0.097n = 7 + m$.

Besides, we have $0 \leq m < 1$, too. So we get $7 \leq 7 + m < 8$.

Thus, we get $7 \leq 0.097n < 8 \Rightarrow \frac{7}{0.097} \leq n < \frac{8}{0.097} \Rightarrow 72 + \frac{16}{97} \leq n < 82 + \frac{46}{97}$.

Therefore, $n = 73, 74, 75, \ldots,$ and 82.

Of course, we can begin with setting the relational expression as follows.

$\mathbf{10^7 \le 1.25^{\mathit{n}} < 10^8}$. How?

Since the number of digits above the decimal point in $\mathbf{1.25^{\mathit{n}}}$ is 8, the highest place value in it is $\mathbf{10^7}$.

Thus, we can set $\mathbf{10^7 \le 1.25^{\mathit{n}} < 10^8}$.

Then, we can take the common log of each term in the expression above, and then, do the log algebra. So taking the logs, then doing the algebra, we get

$$\mathbf{10^7 \le 1.25^{\mathit{n}} < 10^8 \Rightarrow \log 10^7 \le \log 1.25^{\mathit{n}} < \log 10^8 \Rightarrow 7 \le \log 1.25^{\mathit{n}} < 8}$$
$$\mathbf{\Rightarrow 7 \le \mathit{n} \log 1.25 < 8}.$$

Next, gettig the value of $\mathbf{\log 1.25}$, we've got to do some more log algebra, and doing it, we can get

$$\mathbf{\log 1.25 = \log \tfrac{125}{100} = \log \tfrac{1000}{800} = \log \tfrac{10}{8} = 1 - 3\log 2 = 1 - 3 \cdot 0.3010 = 1 - 0.9030 = 0.097}.$$

Thus, we get $\mathbf{7 \le 0.097\mathit{n} < 8 \Rightarrow \tfrac{7}{0.097} \le \mathit{n} < \tfrac{8}{0.097} \Rightarrow 72 + \tfrac{16}{97} \le \mathit{n} < 82 + \tfrac{46}{97}}$.

Therefore, $\mathbf{\mathit{n} = 73, 74, 75, \ldots,}$ and $\mathbf{82}$.

In short:

$\mathbf{\log 1.25^{\mathit{n}} = \mathit{n} \log 1.25 = 7 + \mathit{m}}$, where $\mathbf{0 \le \mathit{m} < 1}$.

$$\mathbf{\log 1.25 = \log \tfrac{125}{100} = \log \tfrac{1000}{800} = \log \tfrac{10}{8} = 1 - 3\log 2 = 1 - 3 \cdot 0.3010 = 1 - 0.9030 = 0.097}.$$

Thus, $\mathbf{\mathit{n} \log 1.25 = 0.097\mathit{n} = 7 + \mathit{m}}$. Besides, we have $\mathbf{0 \le \mathit{m} < 1}$, too.

Thus, $\mathbf{7 \le 7 + \mathit{m} < 8 \Rightarrow 7 \le 0.097\mathit{n} < 8 \Rightarrow \tfrac{7}{0.097} \le \mathit{n} < \tfrac{8}{0.097} \Rightarrow 72 + \tfrac{16}{97} \le \mathit{n} < 82 + \tfrac{46}{97}}$.

Therefore, $\mathbf{\mathit{n} = 73, 74, 75, \ldots,}$ and $\mathbf{82}$.

Suggestions or Solutions
To the Problem 0 in the Example 1

Assuming that *a* and *b* are positive integers, and that a^{100} has 120 digits, find the highest place value in a number, a^{-1}.

$\log a = 119 + m$, and $0 \leq m < 1$.

$0 \leq m < 1 \Rightarrow 119 \leq 119 + m < 120 \Rightarrow 119 \leq 100 \log a < 120 \Rightarrow 1.19 \leq \log a < 1.2$

$\Rightarrow -1.2 < -\log a \leq -1.19 \Rightarrow -1.2 < \log a^{-1} \leq -1.19 \Rightarrow -1 - 0.2 < \log a^{-1} \leq -1 - 0.19$

$\Rightarrow -2 + 0.8 < \log a^{-1} \leq -2 + 0.81.$

So the characteristic is -2, and therefore, the highest place value is 10^{-2}.

If not quite sure of the idea behind the processes above, follow the steps below.

Since a^{100} has 120 digits, we can set $\log a^{100} = 119 + m$, where $0 \leq m < 1$.

Thus, we get $100 \log a = 119 + m$, where $0 \leq m < 1$.

Now, we've got to do some log algebra. What then do we begin the algebra with?

We have $0 \leq m < 1$, so let's begin with it.

$0 \leq m < 1 \Rightarrow 119 \leq 119 + m < 120 \Rightarrow 119 \leq 100 \log a < 120 \Rightarrow 1.19 \leq \log a < 1.2$
$\Rightarrow -1.2 < -\log a \leq -1.19 \Rightarrow -1.2 < \log a^{-1} \leq -1.19 \Rightarrow -1 - 0.2 < \log a^{-1} \leq -1 - 0.19$.

Why do we do it that way, though?

That's because the mantissa is positive, and we want to get the characteristic, too.

Thus, getting the characteristic, too, we get

$\mathbf{-1 - 1 + 1 - 0.2 < \log \mathit{a}^{-1} \le -1 - 1 + 1 - 0.19 \Rightarrow -2 + 0.8 < \log \mathit{a}^{-1} \le -2 + 0.81}$.

So the characteristic is -2.

Therefore, the highest place value in a^{-1} is 10^{-2}.

In other words, it has one zero before the first nonzero digit appears below the decimal point.

That is, the first nonzero digit appears in the second digit below the point.

In short:

$\mathbf{\log \mathit{a} = 119 + \mathit{m}}$, and $\mathbf{0 \le \mathit{m} < 1}$.

$\mathbf{0 \le \mathit{m} < 1 \Rightarrow 119 \le 119 + \mathit{m} < 120 \Rightarrow 119 \le 100 \log \mathit{a} < 120 \Rightarrow 1.19 \le \log \mathit{a} < 1.2}$

$\mathbf{\Rightarrow -1.2 < -\log \mathit{a} \le -1.19 \Rightarrow -1.2 < \log \mathit{a}^{-1} \le -1.19 \Rightarrow -1 - 0.2 < \log \mathit{a}^{-1} \le -1 - 0.19}$

$\mathbf{\Rightarrow -2 + 0.8 < \log \mathit{a}^{-1} \le -2 + 0.81.}$

So the characteristic is -2, and therefore, the highest place value is 10^{-2}.

Suggestions or Solutions
To the Problem 1 in the Example 1

Assuming that a and b are positive integers, and that a^{100} has 120 digits, find the value of b that makes a^b a 10-digit integer.

$\log a^b = b \log a = 9 + m$, where $0 \le m < 1$.

$0 \le m < 1 \Rightarrow 9 \le 9 + m < 10 \Rightarrow 9 \le b \log a < 10$.

$100 \log a = 119 + m$ where, $0 \le m < 1$.

$0 \le m < 1 \Rightarrow 119 \le 119 + m < 120 \Rightarrow 119 \le 100 \log a < 120 \Rightarrow 1.19 \le \log a < 1.2$
$\Rightarrow 1.19b \le b \log a < 1.2b$.

Besides, $9 \le b \log a < 10$, too. So $9 \le 1.19b \le b \log a < 1.2b \le 10.$

Then, first, we get $9 \le 1.19b \Rightarrow \frac{9}{1.19} \le b \Rightarrow \frac{9}{1.19} = \frac{900}{119} = 7 + \frac{67}{119} \le b$.

Next, we get $1.2b \le 10 \Rightarrow b \le \frac{10}{1.2} = \frac{100}{12} = 8 + \frac{4}{12} = 8 + \frac{1}{3}$.

So we get $7 + \frac{67}{119} \le b \le 8 + \frac{1}{3} \Rightarrow b = 8$, since b is an integer.

If not quite sure of the idea behind the processes above, follow the steps below.

Since a^b is a 10-digit integer, the highest place value in it is 10^9.

Thus, we can set $\log a^b = 9 + m$, where $0 \le m < 1$, and we get $b \log a = 9 + m$.

So we can get the extent of $b \log a$.   How?

We have $0 \le m < 1$, so we can get $0 \le m < 1 \Rightarrow 9 \le 9 + m < 10 \Rightarrow 9 \le b \log a < 10$.

Now, we need the value of $\mathbf{\log \boldsymbol{a}}$.   How then can we get it?

Since $\boldsymbol{a}^{100}$ has 120 digits, we can set $\mathbf{100 \log \boldsymbol{a} = 119 + \boldsymbol{m}}$, where $\mathbf{0 \le \boldsymbol{m} < 1}$. So we get

$$\mathbf{0 \le \boldsymbol{m} < 1 \Rightarrow 119 \le 119 + \boldsymbol{m} < 120 \Rightarrow 119 \le 100 \log \boldsymbol{a} < 120 \Rightarrow 1.19 \le \log \boldsymbol{a} < 1.2}$$

Thus, we get $\mathbf{1.19\boldsymbol{b} \le \boldsymbol{b} \log \boldsymbol{a} < 1.2\boldsymbol{b}}$.

Now, we have $\mathbf{9 \le \boldsymbol{b} \log \boldsymbol{a} < 10}$, and $\mathbf{1.19\boldsymbol{b} \le \boldsymbol{b} \log \boldsymbol{a} < 1.2\boldsymbol{b}}$.

So we need to have $\mathbf{9 \le 1.19\boldsymbol{b}}$, and $\mathbf{1.2\boldsymbol{b} \le 10}$.

In sum, we have $\mathbf{9 \le 1.19\boldsymbol{b} \le \boldsymbol{b} \log \boldsymbol{a} < 1.2\boldsymbol{b} \le 10}$.

Thus, first, we get $\mathbf{9 \le 1.19\boldsymbol{b} \Rightarrow \frac{9}{1.19} \le \boldsymbol{b} \Rightarrow \frac{9}{1.19} = \frac{900}{119} = 7 + \frac{67}{119} \le \boldsymbol{b}}$.

Next, we get $\mathbf{1.2\boldsymbol{b} \le 10 \Rightarrow \boldsymbol{b} \le \frac{10}{1.2} = \frac{100}{12} = 8 + \frac{4}{12} \Rightarrow \boldsymbol{b} \le 8 + \frac{1}{3}}$.

Thus, we get $\mathbf{7 + \frac{67}{119} \le \boldsymbol{b} \le 8 + \frac{1}{3} \Rightarrow \boldsymbol{b} = 8}$, since $\boldsymbol{b}$ is an integer.