Examples 9 on Logarithms

Click this to start.

Examples 9 on Logarithms

0. Assuming $\dfrac{x(y+z-x)}{\log_a x} = \dfrac{y(z+x-y)}{\log_a y} = \dfrac{z(x+y-z)}{\log_a z}$, show this: $y^z z^y = z^x x^z = x^y y^x$.

1. Assuming $\log_{2a} a = x$, and $\log_{3a} 2a = y$, show this: $2^{1-xy} = 3^{y-xy}$.

2. Assuming $3^x = 4^y = 6^z$, show this: $\dfrac{1}{2y} = \dfrac{1}{z} - \dfrac{1}{x}$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming $\dfrac{x(y+z-x)}{\log_a x} = \dfrac{y(z+x-y)}{\log_a y} = \dfrac{z(x+y-z)}{\log_a z}$, **show this:** $y^z z^y = z^x x^z = x^y y^x$.

Setting $\dfrac{x(y+z-x)}{\log_a x} = \dfrac{y(z+x-y)}{\log_a y} = \dfrac{z(x+y-z)}{\log_a z} = \dfrac{1}{m}$, we get

$\log_a x = mx(y+z-x)$, $\log_a y = my(z+x-y)$, and $\log_a z = mz(x+y-z)$.

So we get $\log_a x = mx(y+z-x) \Rightarrow x = a^{mx(y+z-x)}$. Thus, we get

$x^z = (a^{mx(y+z-x)})^z = a^{mxz(y+z-x)}$, and $x^y = (a^{mx(y+z-x)})^y = a^{mxy(y+z-x)}$.

We know the same is true for y and z, too, since they are in the same situation as x is in. Thus, we get these:

$y^z = (a^{my(z+x-y)})^z = a^{myz(z+x-y)}$, and $y^x = (a^{my(z+x-y)})^x = a^{myx(z+x-y)}$.

$z^x = (a^{mz(x+y-z)})^x = a^{mzx(x+y-z)}$, and $z^y = (a^{mz(x+y-z)})^y = a^{mzy(x+y-z)}$.

So we get $x^y y^x = a^{mxy(y+z-x)} a^{myx(z+x-y)} = a^{mxy(y+z-x)+myx(z+x-y)}$.

Looking at the exponent part only, we can see this:

$mxy(y+z-x) + myx(z+x-y) = mxyy + mxyz - mxyx + mxyz + myxx - myxy = 2mxyz.$

Thus, we get $x^y y^x = a^{2mxyz}$.

Since all the variables are in the same situation, the same is true for $y^z z^y$ and $z^x x^z$, too.

Thus, we get $y^z z^y = a^{2mxyz}$, and $z^x x^z = a^{2mxyz}$.

Therefore, $x^y y^x = y^z z^y = z^x x^z$.

If not quite sure of the idea behind the processes above, follow the steps below.

This is just another example on log algebra, too. So we have nothing to memorize here, and we will just get more used to manipulations of algebra with logs.

Examining the equality we are to prove, we can see that it is in terms of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ only with no log at all. Thus, we may want to extract $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ from the assumption.

Besides, we can notice that all the variables are in the same situation.

Thus, we may want to concentrate one of them, first, since the same idea will probably apply to the others. Where do we begin, though?

We can put the problem this way, too:

$$\frac{x(y+z-x)}{\log_a x}=\frac{y(z+x-y)}{\log_a y}=\frac{z(x+y-z)}{\log_a z} \Rightarrow y^z z^y = z^x x^z = x^y y^x.$$

So we may want to begin with the assumption given.

What have we got in the assumption, though?

We've got just a bunch of complicated expressions connected by equal signs, don't we?

So it looks quite messy. It's not too bad, though. It seems quite involved, but eventually, will reduce to a ratio. In fact, it is nothing but a ratio. So we've got a ratio to work with.

We can simplify it to be in a form of $\frac{A}{B}=\frac{C}{D}=\frac{E}{F}$. Why such a form, though?

It says all the three ratios are the same.

It doesn't specify what they are same to be, though. Then, what do we do?

Such a ratio as above can be taken for a number as 3, 0.5, 8, 1.2, -1, etc.

For instance, we can have $\frac{2}{4} = \frac{1}{2} = 0.5$, $\frac{-6}{4} = \frac{-3}{2} = -1.5$, $\frac{12}{4} = \frac{36}{12} = 3$, etc.

So we can set to a number the ratio above so that we can refer to the ratio more readily.

We don't know what the number is, though. Let's then, set it to a constant m.

So suppose now, $\frac{x(y+z-x)}{\log_a x} = \frac{y(z+x-y)}{\log_a y} = \frac{z(x+y-z)}{\log_a z} = m$.

Then, we may want to expand a little bit the ratio above to see if such a setting will let us readily extract x, y, and z. So expanding the first part above, we get

$$\frac{x(y+z-x)}{\log_a x} = m \Rightarrow m \log_a x = x(y+z-x) \Rightarrow \log_a x = \frac{x}{m}(y+z-x).$$

Then, m is in the denominator, and thus, can cause further calculations to be messier.

So we may want to set the ratio differently, and for instance, we can try $\frac{1}{m}$, where $m \neq 0$, of course. What if it doesn't work, either?

Then, we want to try something else, of course.

Now, setting $\frac{x(y+z-x)}{\log_a x} = \frac{y(z+x-y)}{\log_a y} = \frac{z(x+y-z)}{\log_a z} = \frac{1}{m}$, we get

$\log_a x = mx(y+z-x)$, $\log_a y = my(z+x-y)$, and $\log_a z = mz(x+y-z)$.

Let's focus on x only, for now.

Then, we get $\log_a x = mx(y+z-x) \Rightarrow x = a^{mx(y+z-x)}$ by the definition for logs.

Now, we have x^z and x^y in the equality where $y^z z^y = z^x x^z = x^y y^x$.

So we may want to make it that way. Then, we get these:

$x = a^{mx(y + z - x)} \Rightarrow x^z = (a^{mx(y + z - x)})^z = a^{mxz(y + z - x)}$.

$x = a^{mx(y + z - x)} \Rightarrow x^y = (a^{mx(y + z - x)})^y = a^{mxy(y + z - x)}$.

We know the same is true for y and z, too, since they are in the same situation as x is in. Thus, we get these:

$y^z = (a^{my(z + x - y)})^z = a^{myz(z + x - y)}$.

$y^x = (a^{my(z + x - y)})^x = a^{myx(z + x - y)}$.

$z^x = (a^{mz(x + y - z)})^x = a^{mzx(x + y - z)}$.

$z^y = (a^{mz(x + y - z)})^y = a^{mzy(x + y - z)}$.

Now, all we have to do is to make each of $y^z z^y$, $z^x x^z$, and $x^y y^x$, and show that they are all equal to each other.

To begin with, we get $x^y y^x = a^{mxy(y + z - x)} a^{myx(z + x - y)} = a^{mxy(y + z - x) + myx(z + x - y)} = a^{2mxyz}$. How?

Looking at the exponent part only, we can see this:

$mxy(y + z - x) + myx(z + x - y) = mxyy + mxyz - mxyx + mxyz + myxx - myxy = 2mxyz$.

Since all the variables are in the same situation, the same is true for $y^z z^y$ and $z^x x^z$, too.

Thus, we get $y^z z^y = a^{2mxyz}$, and $z^x x^z = a^{2mxyz}$.

Therefore, $x^y y^x = y^z z^y = z^x x^z$.

In short:

Setting $\frac{x(y+z-x)}{\log_a x} = \frac{y(z+x-y)}{\log_a y} = \frac{z(x+y-z)}{\log_a z} = \frac{1}{m}$, we get

$\log_a x = mx(y + z - x)$, $\log_a y = my(z + x - y)$, and $\log_a z = mz(x + y - z)$.

So we get $\log_a x = mx(y + z - x) \Rightarrow x = a^{mx(y + z - x)}$. Thus, we get

$x^z = (a^{mx(y + z - x)})^z = a^{mxz(y + z - x)}$, and $x^y = (a^{mx(y + z - x)})^y = a^{mxy(y + z - x)}$.

We know the same is true for y and z, too, since they are in the same situation as x is in. Thus, we get these:

$y^z = (a^{my(z + x - y)})^z = a^{myz(z + x - y)}$, and $y^x = (a^{my(z + x - y)})^x = a^{myx(z + x - y)}$.

$z^x = (a^{mz(x + y - z)})^x = a^{mzx(x + y - z)}$, and $z^y = (a^{mz(x + y - z)})^y = a^{mzy(x + y - z)}$.

So we get $x^y y^x = a^{mxy(y + z - x)} a^{myx(z + x - y)} = a^{mxy(y + z - x) + myx(z + x - y)}$.

Looking at the exponent part only, we can see this:

$mxy(y + z - x) + myx(z + x - y) = mxyy + mxyz - mxyx + mxyz + myxx - myxy = 2mxyz.$

Thus, we get $x^y y^x = a^{2mxyz}$.

Since all the variables are in the same situation, the same is true for $y^z z^y$ and $z^x x^z$, too.

Thus, we get $y^z z^y = a^{2mxyz}$, and $z^x x^z = a^{2mxyz}$.

Therefore, $x^y y^x = y^z z^y = z^x x^z$.

Suggestions or Solutions
To the Problem in the Example 1

Assuming $\log_{2a} a = x$, and $\log_{3a} 2a = y$, show that $2^{1-xy} = 3^{y-xy}$.

$x = \log_{2a} a = \dfrac{\log a}{\log 2a}$, and $y = \log_{3a} 2a = \dfrac{\log 2a}{\log 3a}$.

Thus, $xy = \dfrac{\log a}{\log 3a} = \log_{3a} a$.

$1 - xy = \log_{3a} 3a - \log_{3a} a = \log_{3a} \frac{3a}{a} = \log_{3a} 3 \Rightarrow 3 = (3a)^{1-xy}$.

$y - xy = \log_{3a} 2a - \log_{3a} a = \log_{3a} \frac{2a}{a} = \log_{3a} 2 \Rightarrow 2 = (3a)^{y-xy}$.

$3 = (3a)^{1-xy} \Rightarrow 3^{y-xy} = \{(3a)^{1-xy}\}^{y-xy}$.

$2 = (3a)^{y-xy} \Rightarrow 2^{1-xy} = \{(3a)^{y-xy}\}^{1-xy} = \{(3a)^{1-xy}\}^{y-xy}$.

Therefore, $2^{1-xy} = 3^{y-xy}$.

If not quite sure of the idea behind the processes above, follow the steps below.

This is just another example on log algebra, too.

Looking at the equality we are to prove, we see no log, but can see that both sides are antilogs, where the bases are two numbers, 2 and 3, but the exponents are expressions in terms of x and y.

Examining the assumption though, we can see x and y are all expressed in terms of logs.

In such cases, we usually remove the log signs, extract the bases necessary, which are 2 and 3, and then, form the equality, which is to be proved, of course.

Anyhow, log signs will probably disappear during such a forming process.

So let's now, begin extracting the two bases, 2 and 3.

We know a log can be put in a fractional form.
Quite often, we in fact, take a log for a fraction, where a numerator is over a denominator.

Thus, we may want to begin with putting the two logs in fractional forms. Then, we get

$x = \log_{2a} a = \dfrac{\log a}{\log 2a}$, and $y = \log_{3a} 2a = \dfrac{\log 2a}{\log 3a}$.

Thus, we get $xy = \dfrac{\log a}{\log 2a} \cdot \dfrac{\log 2a}{\log 3a} = \dfrac{\log a}{\log 3a} = \log_{3a} a$.

So we can now try first, forming the exponents in the equality, $2^{1-xy} = 3^{y-xy}$.

$1 - xy = \log_{3a} 3a - \log_{3a} a = \log_{3a} \frac{3a}{a} = \log_{3a} 3 \Rightarrow 3 = (3a)^{1-xy}$.

$y - xy = \log_{3a} 2a - 1\mathrm{og}_{3a}\, a = \log_{3a} \frac{2a}{a} = \log_{3a} 2 \Rightarrow 2 = (3a)^{y-xy}$.

Let's next, form each side in the equality.   Then, we get these:

$3 = (3a)^{1-xy} \Rightarrow 3^{y-xy} = \{(3a)^{1-xy}\}^{y-xy}$.

$2 = (3a)^{y-xy} \Rightarrow 2^{1-xy} = \{(3a)^{y-xy}\}^{1-xy} = \{(3a)^{1-xy}\}^{y-xy}$, since we have an identity where $B^{mn} = B^{nm}$.

Thus, we can see this: $2^{1-xy} = 3^{y-xy}$.

Suggestions or Solutions
To the Problem in the Example 2

Assuming $3^x = 4^y = 6^z$, show that $\frac{1}{2y} = \frac{1}{z} - \frac{1}{x}$.

Setting $3^x = 4^y = 6^z = k$ and applying logs to it, we get $\log k = x \log 3 = y \log 4 = z \log 6$.

Then, $x = \frac{\log k}{\log 3}$, $y = \frac{\log k}{\log 4}$, and $z = \frac{\log k}{\log 6}$.

Thus, $\frac{1}{x} = \frac{\log 3}{\log k}$, $\frac{1}{2y} = \frac{\log 4}{2\log k} = \frac{\log 2}{\log k}$, and $\frac{1}{z} = \frac{\log 6}{\log k}$.

So $\frac{1}{z} - \frac{1}{x} = \frac{\log 6}{\log k} - \frac{\log 3}{\log k} = \frac{\log 6 - \log 3}{\log k} = \frac{\log \frac{6}{3}}{\log k} = \frac{\log 2}{\log k} = \frac{1}{2y}$.

Therefore, $\frac{1}{2y} = \frac{1}{z} - \frac{1}{x}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Forming the equality, $\frac{1}{2y} = \frac{1}{z} - \frac{1}{x}$, which is to be proved, we need to get x, y, and z.

We can get those from the assumption. How then do we get them?

Apply logs to the equality in the assumption.

Before taking logs of the terms in the equality however, we may want to set the equality equal to a constant so that we can readily extract x, y, and z.

Let's now, set $3^x = 4^y = 6^z = k$, and apply logs to this equaliy.

Then, we get $\log k = \log 3^x = \log 4^y = \log 6^z$.

Thus, we get $\log k = x \log 3 = y \log 4 = z \log 6$.

So we get $x = \frac{\log k}{\log 3}$, $y = \frac{\log k}{\log 4}$, and $z = \frac{\log k}{\log 6}$.

Let's next, form the terms in the equality to be proved.

Then, we get $\frac{1}{x} = \frac{\log 3}{\log k}$, $2y = \frac{2\log k}{\log 4} \Rightarrow \frac{1}{2y} = \frac{\log 4}{2\log k}$, and $\frac{1}{z} = \frac{\log 6}{\log k}$.

Thus, we get $\frac{1}{z} - \frac{1}{x} = \frac{\log 6}{\log k} - \frac{\log 3}{\log k} = \frac{\log 6 - \log 3}{\log k} = \frac{\log \frac{6}{3}}{\log k} = \frac{\log 2}{\log k} \Rightarrow \frac{1}{z} - \frac{1}{x} = \frac{\log 2}{\log k}$.

Meanwhile, $\frac{1}{2y} = \frac{\log 4}{2\log k} = \frac{\log 2^2}{2\log k} = \frac{2\log 2}{2\log k} = \frac{\log 2}{\log k}$.

Therefore, we can see this: $\frac{1}{2y} = \frac{1}{z} - \frac{1}{x}$.

In short:

Setting $\mathbf{3^x = 4^y = 6^z = k}$ and applying logs to it, we get $\mathbf{\log k = x \log 3 = y \log 4 = z \log 6}$.

Then, $x = \frac{\log k}{\log 3}$, $y = \frac{\log k}{\log 4}$, and $z = \frac{\log k}{\log 6}$.

Thus, $\frac{1}{x} = \frac{\log 3}{\log k}$, $\frac{1}{2y} = \frac{\log 4}{2\log k} = \frac{\log 2}{\log k}$, and $\frac{1}{z} = \frac{\log 6}{\log k}$.

So $\frac{1}{z} - \frac{1}{x} = \frac{\log 6}{\log k} - \frac{\log 3}{\log k} = \frac{\log 6 - \log 3}{\log k} = \frac{\log \frac{6}{3}}{\log k} = \frac{\log 2}{\log k} = \frac{1}{2y}$.

Therefore, $\frac{1}{2y} = \frac{1}{z} - \frac{1}{x}$.