Examples A on Logarithms

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0. Assuming $\mathbf{11.2}^{a} = \mathbf{0.0112}^{b} = \mathbf{1000}$, show this: $\frac{1}{a} - \frac{1}{b} = 1$.

1. Solve the equation as follows. $\log \frac{x-2}{2} = \frac{\log x - \log 2}{2}$.

2. Suppose that a, b, and $c > 0$, but $\neq 1$. Suppose also, that $c = ab$, and that $a^{x} = b^{y} = c^{z}$. Then, show this: $z = \frac{xy}{x+y}$.

3. Assuming $9^{a+1} = 45$, $\log_{7} 3 = \frac{1}{b}$, and $\left(\frac{343}{45}\right)^{t} = 105$, find t, and then, put t in terms of a and b.

Suggestions or Solutions
To the Problem in the Example 0

Assuming $11.2^a = 0.0112^b = 1000$, show this: $\frac{1}{a} - \frac{1}{b} = 1$.

We can put the problem in this way, too: $11.2^a = 0.0112^b = 1000 \Rightarrow \frac{1}{a} - \frac{1}{b} = 1$.

In other words, "By means of $11.2^a = 0.0112^b = 1000$, show $\frac{1}{a} - \frac{1}{b} = 1$."

So we get to show the equality $\frac{1}{a} - \frac{1}{b} = 1$ using a fact that $11.2^a = 0.0112^b = 1000$. How?

We can extract a and b from the fact. Then, we put them into the right hand side of the equality, and do the algebra. How do we extract them, though?

Applying logs to the fact, we get

$$\log 11.2^a = \log 0.0112^b = \log 1000 \Rightarrow a \log 11.2 = b \log 0.0112 = \log 1000 = \log 10^3 = 3.$$

So we get $a = \frac{3}{\log 11.2} \Rightarrow \frac{1}{a} = \frac{\log 11.2}{3}$, and $b = \frac{3}{\log 0.0112} \Rightarrow \frac{1}{b} = \frac{\log 0.0112}{3}$.

Thus, $\frac{1}{a} - \frac{1}{b} = \frac{\log 11.2}{3} - \frac{\log 0.0112}{3} = \frac{(\log 11.2 - \log 0.0112)}{3} = \frac{\log \frac{11.2}{0.0112}}{3} = \frac{\log 1000}{3} = 1$.

Note that $\frac{1}{a} - \frac{1}{b} = 1 \Leftrightarrow b - a = ab$, where a and $b \neq 0$.

Thus, we can show $b - a = ab$ instead if it is easier.

Now, what if we have $\mathbf{23^{\mathit{a}} = 0.0023^{\mathit{b}} = 10000}$ instead of $\mathbf{11.2^{\mathit{a}} = 0.0112^{\mathit{b}} = 1000}$?

We still get $\frac{1}{a} - \frac{1}{b} = \mathbf{1}$.

How about $\mathbf{3.5^{\mathit{a}} = 350^{\mathit{b}} = 100}$?

We still get $\frac{1}{a} - \frac{1}{b} = \mathbf{1}$, too.

What then, if $\mathbf{12.5^{\mathit{a}} = 1.25^{\mathit{b}} = 10}$?

We get $\frac{1}{a} - \frac{1}{b} = \mathbf{1}$, still. Noticed the pattern?

Suggestions or Solutions
To the Problem in the Example 1

Solve the following equation. $\log\frac{x-2}{2}=\dfrac{\log x-\log 2}{2}$.

$\log\frac{x-2}{2}=\frac{\log x-\log 2}{2}=\frac{1}{2}\log\frac{x}{2}\Rightarrow 2\log\frac{x-2}{2}=\log\frac{x}{2}\Rightarrow\left(\frac{x-2}{2}\right)^2=\frac{x}{2}$

$\Rightarrow (x-2)^2=2x\Rightarrow x^2-4x+4=2x\Rightarrow x^2-6x+4=0\Rightarrow x=3\pm\sqrt{5}$.

Both $x=3+\sqrt{5}$ and $x=3-\sqrt{5}>0$, so both are allowed for log x.

$x=3-\sqrt{5}\Rightarrow\frac{x-2}{2}=\frac{3-\sqrt{5}-2}{2}=\frac{1-\sqrt{5}}{2}<0$, so $x=3-\sqrt{5}$ is not allowed for $\log\frac{x-2}{2}$.

$x=3+\sqrt{5}\Rightarrow\frac{x-2}{2}=\frac{3+\sqrt{5}-2}{2}=\frac{1+\sqrt{5}}{2}>0$, so $x=3+\sqrt{5}$ is allowed for $\log\frac{x-2}{2}$.

Therefore, $x=3+\sqrt{5}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Such an equation as above can be called a log equation.

Hardly, we can directly solve log equations.

Normally, we remove log signs first, or begin with a substitution.
In this problem, we can remove log signs first.

Removing log signs in this case, we can get an ordinary algebraic equation for x.

$$\log\frac{x-2}{2}=\frac{\log x-\log 2}{2}\Rightarrow 2\log\frac{x-2}{2}=\log x-\log 2\Rightarrow\log\left(\frac{x-2}{2}\right)^2=\log\frac{x}{2}\Rightarrow\left(\frac{x-2}{2}\right)^2=\frac{x}{2}.$$

Of course, we can put it this way, too:

$$\frac{\log x-\log 2}{2}=\frac{1}{2}(\log x-\log 2)=\frac{1}{2}\log\frac{x}{2}=\log\sqrt{\frac{x}{2}}.$$

Thus, we get

$$\log\frac{x-2}{2}=\frac{\log x-\log 2}{2}=\log\sqrt{\frac{x}{2}}\Rightarrow\frac{x-2}{2}=\sqrt{\frac{x}{2}}\Rightarrow\frac{(x-2)^2}{4}=\frac{x}{2}.$$

So the last equation above is just a quadratic equation, yet after solving the equation, we want to check to see if all the roots are OK.

That is because negative cases can get involved due to the squaring operation done at the stage where $\frac{x-2}{2}=\sqrt{\frac{x}{2}}$.

Expanding the left hand side in $\frac{(x-2)^2}{4}=\frac{x}{2}$, we get $\frac{x^2-4x+4}{4}$.

Thus, we get $\frac{x^2-4x+4}{4}=\frac{x}{2}\Rightarrow x^2-4x+4=2x\Rightarrow x^2-6x+4=0$.

The left hand side doesn't simply get factorized to a form of $(x-a)(x-b)$.

So using the quadratic formula, we get $x=3\pm\sqrt{5}$.

Since both $3+\sqrt{5}$ and $3-\sqrt{5}>0$, they have no problem with $\log x$. However, we want to make sure if both are fine with $\log\frac{x-2}{2}$, too.

$x=3+\sqrt{5}\Rightarrow\log\frac{x-2}{2}=\log\frac{3+\sqrt{5}-2}{2}=\log\frac{1+\sqrt{5}}{2}$, which means it's OK, because $\frac{1+\sqrt{5}}{2}>0$.

$x=3-\sqrt{5}\Rightarrow\log\frac{x-2}{2}=\log\frac{3-\sqrt{5}-2}{2}=\log\frac{1-\sqrt{5}}{2}$, which means it's not OK, because $\frac{1-\sqrt{5}}{2}<0$.

Therefore, the solution is $x=3+\sqrt{5}$.

Suggestions or Solutions
To the Problem in the Example 2

Suppose that a, b, and $c > 0$, but $\neq 1$. Suppose also, that $c = ab$, and that $a^x = b^y = c^z$.

Then, show this: $z = \dfrac{xy}{x+y}$**.**

Setting first, $a^x = b^y = c^z = k$, we get

$\log a^x = \log b^y = \log c^z = \log k \Rightarrow x \log a = y \log b = z \log c = \log k$.

Thus, we get $x = \dfrac{\log k}{\log a}$, $y = \dfrac{\log k}{\log b}$, and $z = \dfrac{\log k}{\log c}$.

So we get $xy = \dfrac{\log k}{\log a} \cdot \dfrac{\log k}{\log b} = \dfrac{(\log k)^2}{(\log a)(\log b)}$.

And $x + y = \dfrac{\log k}{\log a} + \dfrac{\log k}{\log b} = \log k \dfrac{(\log b + \log a)}{(\log a)(\log b)} = \dfrac{(\log k)(\log ab)}{(\log a)(\log b)}$.

Thus, we get $\dfrac{xy}{x+y} = \dfrac{(\log k)^2}{(\log a)(\log b)} \cdot \dfrac{(\log a)(\log b)}{(\log k)(\log ab)} = \dfrac{\log k}{\log ab}$.

And we have $z = \dfrac{\log k}{\log c}$, and $c = ab$, too.

Thus, we get $z = \dfrac{\log k}{\log c} = \dfrac{\log k}{\log ab}$, so we get $z = \dfrac{xy}{x+y}$.

If not quite sure of the idea behind the processes above, follow the steps below.

What we want to show is of course, under the assumption that a, b, and $c > 0$, but $\neq 1$.

And we get to show the equality $\boldsymbol{z} = \frac{xy}{x+y}$ using a fact that $\boldsymbol{a^x = b^y = c^z}$ where $\boldsymbol{c = ab}$. How?

We can put each of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Then, we put each of them into the equality, and do the algebra. How do we get $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, though?

We don't actually get $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, themselves. What then do we get?

We can extract logs of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ from the fact given, which is the equality $\boldsymbol{a^x = b^y = c^z}$, then we put each of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ in terms of the logs. So let's extract them now.

We may want to first, set $\boldsymbol{a^x = b^y = c^z = k}$ so that we can readily refer to the fact.
Then, we can extract them applying logs to the fact. So applying logs to the fact, we get

$$\log a^x = \log b^y = \log c^z = \log k \Rightarrow x \log a = y \log b = z \log c = \log k.$$

Thus, we get $\boldsymbol{x} = \dfrac{\log k}{\log a}$, $\boldsymbol{y} = \dfrac{\log k}{\log b}$, and $\boldsymbol{z} = \dfrac{\log k}{\log c}$.

Now, we can put each of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ into the equality where $\boldsymbol{z} = \frac{xy}{x+y}$.

To begin with, we get $\boldsymbol{xy} = \dfrac{\log k}{\log a} \cdot \dfrac{\log k}{\log b} = \dfrac{(\log k)^2}{(\log a)(\log b)}$.

Next, we get

$$x + y = \frac{\log k}{\log a} + \frac{\log k}{\log b} = \log k\left(\frac{1}{\log a} + \frac{1}{\log b}\right) = \log k\,\frac{(\log b + \log a)}{(\log a)(\log b)} = \frac{(\log k)(\log ab)}{(\log a)(\log b)}.$$

Thus, we get $\dfrac{xy}{x+y} = xy\dfrac{1}{x+y} = \dfrac{(\log k)^2}{(\log a)(\log b)} \cdot \dfrac{(\log a)(\log b)}{(\log k)(\log ab)} = \dfrac{\log k}{\log ab}$.

And we have $\boldsymbol{z} = \frac{\log k}{\log c}$, and ***c* = *ab***, too.

Thus, we get $\boldsymbol{z} = \frac{\log k}{\log c} = \frac{\log k}{\log ab}$, so we can see this: $\boldsymbol{z} = \frac{xy}{x+y}$.

In short:

Setting first, $\boldsymbol{a^x = b^y = c^z = k}$, we get

$\boldsymbol{\log a^x = \log b^y = \log c^z = \log k \Rightarrow x \log a = y \log b = z \log c = \log k}$.

Thus, we get $\boldsymbol{x} = \frac{\log k}{\log a}$, $\boldsymbol{y} = \frac{\log k}{\log b}$, and $\boldsymbol{z} = \frac{\log k}{\log c}$.

So we get $\boldsymbol{xy} = \frac{\log k}{\log a} \cdot \frac{\log k}{\log b} = \frac{(\log k)^2}{(\log a)(\log b)}$.

And $\boldsymbol{x + y} = \frac{\log k}{\log a} + \frac{\log k}{\log b} = \log k \frac{(\log b + \log a)}{(\log a)(\log b)} = \frac{(\log k)(\log ab)}{(\log a)(\log b)}$.

Thus, we get $\frac{xy}{x+y} = \frac{(\log k)^2}{(\log a)(\log b)} \cdot \frac{(\log a)(\log b)}{(\log k)(\log ab)} = \frac{\log k}{\log ab}$.

And we have $\boldsymbol{z} = \frac{\log k}{\log c}$, and ***c* = *ab***, too.

Thus, we get $\boldsymbol{z} = \frac{\log k}{\log c} = \frac{\log k}{\log ab}$, so we get $\boldsymbol{z} = \frac{xy}{x+y}$.

Suggestions or Solutions
To the Problem in the Example 3

Assuming $9^{a+1} = 45$, $\log_7 3 = \frac{1}{b}$, and $(\frac{343}{45})^t = 105$, find *t*, and then, put *t* in terms of *a* and *b*.

$$\log\left(\tfrac{343}{45}\right)^t = \log 105 \Rightarrow t\log\tfrac{343}{45} = \log 105 \Rightarrow t = \frac{\log 105}{\log\frac{343}{45}} \Rightarrow t = \frac{\log 105}{\log 343 - \log 45}.$$

$$9^{a+1} = 45 \Rightarrow \log 9^{a+1} = \log 45 \Rightarrow (a+1)\log 9 = \log 5 + \log 9 \Rightarrow\ a+1 = \frac{\log 5 + \log 9}{\log 9}$$

$$\Rightarrow a = \frac{\log 5}{\log 9} = \frac{\log 5}{2\log 3} \Rightarrow 2a = \frac{\log 5}{\log 3} = \log_3 5.\ \text{ And } \log_7 3 = \tfrac{1}{b} \Rightarrow \tfrac{1}{\log_7 3} = \log_3 7.$$

And we have **$105 = 3\cdot5\cdot7$**, **$343 = 7^3$**, and **$45 = 3^2 5$**. So we get

$\log 105 = \log(3\cdot5\cdot7) = \log 3 + \log 5 + \log 7$, **$\log 343 = \log 7^3 = 3\log 7$**, and

$\log 45 = \log 5 + 2\log 3$.

Thus, we get $t = \dfrac{\log 105}{\log 343 - \log 45} = \dfrac{\log 3 + \log 5 + \log 7}{3\log 7 - \log 5 - 2\log 3} = \dfrac{\log_3 3 + \log_3 5 + \log_3 7}{3\log_3 7 - \log_3 5 - 2\log_3 3}$

$$= \frac{1 + \log_3 5 + \log_3 7}{3\log_3 7 - \log_3 5 - 2} = \frac{1 + 2a + b}{3b - 2a - 2}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

We can readily find ***t***. Finding ***t*** though, is just one thing, and putting it in terms of ***a*** and ***b*** is another.

Applying logs to the equation for $\boldsymbol{t}$ where $\mathbf{(\frac{343}{45})^{\boldsymbol{t}} = 105}$, we get

$$\mathbf{\log\left(\frac{343}{45}\right)^{\boldsymbol{t}} = \log 105 \Rightarrow \boldsymbol{t}\log\frac{343}{45} = \log 105 \Rightarrow \boldsymbol{t} = \frac{\log 105}{\log\frac{343}{45}} \Rightarrow \boldsymbol{t} = \frac{\log 105}{\log 343 - \log 45}}.$$

And of course, by the definition for logs, we can get $\mathbf{\boldsymbol{t} = \log_{\frac{343}{45}} 105}$, which is however, no other than $\mathbf{\frac{\log 105}{\log 343 - \log 45}}$, because we have $\mathbf{\log_{\frac{343}{45}} 105 = \frac{\log 105}{\log\frac{343}{45}}}$.

Now, we need to put $\boldsymbol{t}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, so we may want to get $\boldsymbol{a}$ and $\boldsymbol{b}$, first. How?

We are given two facts, one is an equation for $\boldsymbol{a}$, and the other is an equation for $\boldsymbol{b}$.

So solving the equations for $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, we can get $\boldsymbol{a}$ and $\boldsymbol{b}$.

Thus, beginning with $\boldsymbol{a}$, we get $\mathbf{9^{\boldsymbol{a}+1} = 45 \Rightarrow \log 9^{\boldsymbol{a}+1} = \log 45}$

$$\mathbf{\Rightarrow (\boldsymbol{a}+1)\log 9 = \log 5 + \log 9 \Rightarrow \boldsymbol{a}+1 = \frac{\log 5+\log 9}{\log 9} \Rightarrow \boldsymbol{a} = \frac{\log 5+\log 9}{\log 9} - 1 = \frac{\log 5}{\log 9} \Rightarrow \boldsymbol{a} = \frac{\log 5}{\log 9}}.$$

Next, moving on to $\boldsymbol{b}$, we get $\mathbf{\log_7 3 = \frac{1}{\boldsymbol{b}} \Rightarrow \boldsymbol{b} = \frac{1}{\log_7 3} = \log_3 7}$, where the base is 3.

So we may want to put $\boldsymbol{a}$ in this way: $\mathbf{\boldsymbol{a} = \frac{\log 5}{\log 9} = \frac{\log 5}{\log 3^2} = \frac{\log 5}{2\log 3} \Rightarrow 2\boldsymbol{a} = \frac{\log 5}{\log 3} = \log_3 5}$.

Of course, we can put it this way, too: $\mathbf{\boldsymbol{a} = \frac{\log 5}{\log 9} = \frac{\log_3 5}{\log_3 9} = \frac{\log_3 5}{2} \Rightarrow 2\boldsymbol{a} = \log_3 5}$.

How do we get $\mathbf{\frac{\log 5}{\log 9} = \frac{\log_3 5}{\log_3 9}}$, though?

We will see how to get it shortly. So let's see for now, what we can do about ***t***.

We have $\boldsymbol{t} = \dfrac{\log 105}{\log 343 - \log 45}$.

So it is made of logs, where the antilogs are 105, 343, and 45.
Thus, we may want to see what the antilogs are composed of. How though?

Factorizing the antilogs, we can readily see the components.

We have $\mathbf{105 = 3 \cdot 5 \cdot 7}$, $\mathbf{343 = 7^3}$, and $\mathbf{45 = 3^2 5}$. So we can see these:

$\mathbf{\log 105 = \log (3 \cdot 5 \cdot 7) = \log 3 + \log 5 + \log 7}$, $\mathbf{\log 343 = \log 7^3 = 3 \log 7}$, and

$\mathbf{\log 45 = \log 5 + 2 \log 3}$.

Thus, we get $\boldsymbol{t} = \dfrac{\log 105}{\log 343 - \log 45} = \dfrac{\log 3 + \log 5 + \log 7}{3 \log 7 - (\log 5 + 2 \log 3)} = \dfrac{\log 3 + \log 5 + \log 7}{3 \log 7 - \log 5 - 2 \log 3}$.

Now, we have $\boldsymbol{t} = \dfrac{\log 3 + \log 5 + \log 7}{3 \log 7 - \log 5 - 2 \log 3}$, $\mathbf{2}\boldsymbol{a} = \mathbf{\log_3 5}$, and $\boldsymbol{b} = \mathbf{\log_3 7}$.

Besides, we have $\dfrac{\log x}{\log y} = \dfrac{\frac{\log x}{\log z}}{\frac{\log y}{\log z}} = \dfrac{\log_z x}{\log_z y}$, too. (So we get $\frac{\log 5}{\log 9} = \frac{\log_3 5}{\log_3 9}$, as well.)

Thus, dividing by $\mathbf{\log 3}$ both the numerator and denominator in $\dfrac{\log 3 + \log 5 + \log 7}{3 \log 7 - \log 5 - 2 \log 3}$,

we get $\boldsymbol{t} = \dfrac{\log_3 3 + \log_3 5 + \log_3 7}{3 \log_3 7 - \log_3 5 - 2 \log_3 3} = \dfrac{1 + \log_3 5 + \log_3 7}{3 \log_3 7 - \log_3 5 - 2}$.

Thus, we get $\boldsymbol{t} = \dfrac{1 + \log_3 5 + \log_3 7}{3 \log_3 7 - \log_3 5 - 2} = \dfrac{1 + 2a + b}{3b - 2a - 2}$.

In short:

$$\log\left(\tfrac{343}{45}\right)^t = \log 105 \Rightarrow t\log\tfrac{343}{45} = \log 105 \Rightarrow t = \frac{\log 105}{\log\tfrac{343}{45}} \Rightarrow t = \frac{\log 105}{\log 343 - \log 45}.$$

$$9^{a+1} = 45 \Rightarrow \log 9^{a+1} = \log 45 \Rightarrow (a+1)\log 9 = \log 5 + \log 9 \Rightarrow \quad a+1 = \frac{\log 5 + \log 9}{\log 9}$$

$$\Rightarrow a = \frac{\log 5}{\log 9} = \frac{\log 5}{2\log 3} \Rightarrow 2a = \frac{\log 5}{\log 3} = \log_3 5.$$ And $\log_7 3 = \tfrac{1}{b} \Rightarrow \tfrac{1}{\log_7 3} = \log_3 7$.

And we have **105 = 3·5·7**, **343 = 7^3**, and **45 = $3^2$5**. So we get

log 105 = log (3·5·7) = log 3 + log 5 + log 7, **log 343 = log 7^3 = 3 log 7**, and

log 45 = log 5 + 2 log 3.

Thus, we get $t = \dfrac{\log 105}{\log 343 - \log 45} = \dfrac{\log 3 + \log 5 + \log 7}{3\log 7 - \log 5 - 2\log 3} = \dfrac{\log_3 3 + \log_3 5 + \log_3 7}{3\log_3 7 - \log_3 5 - 2\log_3 3}$

$$= \frac{1 + \log_3 5 + \log_3 7}{3\log_3 7 - \log_3 5 - 2} = \frac{1 + 2a + b}{3b - 2a - 2}.$$