Examples B on Logarithms

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0.0. Assuming that $(\log xy)^2 = (\log x)(\log y) + \log x - \log y - 1$, find the value of xy.

0.1. Assuming that ab, bc, and $ca > 0$, and that $(\log ac)(\log bc) + 1 = 0$ is an equation for c, find an additional condition on a and b so that the equation has real roots.

1. Suppose an equation for x, $(\log 2x)(\log 3x) = -a^2$ has two real roots. Then:

1.0. Find the extent of a.

1.1. Find the value of the product of the roots.

2. Solve the following equation for x.

$\left(\dfrac{x}{10^{100}}\right)^u = \dfrac{\log 10^x}{x}$ where $u = (\log x^v - \log x^7 + 10)^3$ where $v = \log x$.

Suggestions or Solutions
To the Problem 0 in the Example 0

Assuming that $(\log xy)^2 = (\log x)(\log y) + \log x - \log y - 1$, find the value of xy.

Setting $\boldsymbol{u = \log x}$ and $\boldsymbol{v = \log y}$, we get

$$(\log xy)^2 = (\log x + \log y)^2 = (\log x)(\log y) + \log x - \log y - 1$$
$$\Rightarrow (u + v)^2 = uv + u - v - 1 \Rightarrow u^2 + 2uv + v^2 = uv + u - v - 1$$
$$\Rightarrow u^2 + uv - u + v + v^2 + 1 = 0.$$

Taking the final equation above as an equation for $\boldsymbol{u}$, we can get a quadratic equation as follows. $\boldsymbol{u^2 + (v - 1)u + v^2 + v + 1 = 0}$.

Since the equaion above needs to have real roots, we need to have
$\boldsymbol{D = (v - 1)^2 - 4(v^2 + v + 1) \geq 0}$.

And simplifying the expression above, and rewriting the result, we can get

$$D = v^2 - 2v + 1 - 4v^2 - 4v - 4 = -3v^2 - 6v - 3 = -3(v^2 + 2v + 1) = -3(v + 1)^2 \geq 0$$
$$\Rightarrow v = -1 \Rightarrow \log y = -1.$$

Thus, $\boldsymbol{y = 10^{-1}}$.

Next, $\boldsymbol{v = -1 \Rightarrow u^2 + (v - 1)u + v^2 + v + 1 = u^2 - 2u + 1 = (u - 1)^2 = 0}$.

So $\boldsymbol{u = 1}$, and thus, $\boldsymbol{\log x = 1 \Rightarrow x = 10}$. Therefore, $\boldsymbol{xy = 1}$.

If not quite sure of the idea behind the processes above, follow the steps below.

We have two unknowns, which are $\boldsymbol{x}$ and $\boldsymbol{y}$, so we need to have two equations, but are given one equation only. It's OK, though, in this case. Why?

The equation given is quadratic. Working with a quadratic equation, we often take from the equation a special value or expression, which is intrinsic to an equation quadratic.

And the value or expression is called the discriminant, usually denoted by $\boldsymbol{D}$.

Suppose that we have an equation for $\boldsymbol{x}$ as follows: $\boldsymbol{ax^2 + bx + c = 0}$ where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant. Then, the discriminant $\boldsymbol{D = b^2 - 4ac}$. If in particular, $\boldsymbol{b = 2k}$, that is, $\boldsymbol{b}$ is even, we often set: $\boldsymbol{D/4 = k - ac}$. What then is the discriminant about?

It is in fact, what's inside the square root sign in the quadratic formula, which is below.

$\boldsymbol{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$. Thus, the formula can be put this way, too: $\boldsymbol{x = \dfrac{-b \pm \sqrt{D}}{2a}}$ for short.

And if $\boldsymbol{b = 2k}$, we can put it this way, too: $\boldsymbol{x = \dfrac{-k \pm \sqrt{k^2 - ac}}{a}}$. That's because we get

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2k \pm \sqrt{4k^2 - 4ac}}{2a} = \frac{-2k \pm 2\sqrt{k^2 - ac}}{2a} = \frac{-k \pm \sqrt{k^2 - ac}}{a}}.$$

And we know what's inside the square root sign is ≥ 0 if the root is real.
Therefore, if a quadratic equation has real roots, the discriminant $\boldsymbol{D}$ has to be $\geq \mathbf{0}$.

If $\boldsymbol{D = 0}$, the equation has a double root, which is $\boldsymbol{\dfrac{-b}{2a}}$.

If $\boldsymbol{D < 0}$, the equation has no real root.

And if $\boldsymbol{D > 0}$, the equation has two different real roots, which are $\boldsymbol{\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

Since we have a log equation however, we want to check what's inside the log, too.

That is, we want to consider the requirement on logarithms as follows.

- The *antilog* and the *base* in a log have to be *positive*, but the *base cannot* be 1.

Now, rewriting the equation given, we can have better look at the equation.
And rewriting it, we can begin with, for simplicity, setting $\boldsymbol{u} = \mathbf{log}\ \boldsymbol{x}$, and $\boldsymbol{v} = \mathbf{log}\ \boldsymbol{y}$.

Then, we get $(\log xy)^2 = (\log x + \log y)^2 = (\log x)(\log y) + \log x - \log y - 1$

$\Rightarrow (u + v)^2 = uv + u - v - 1 \Rightarrow u^2 + 2uv + v^2 = uv + u - v - 1$

$\Rightarrow u^2 + uv - u + v + v^2 + 1 = 0$.   What then?

We can now take the last equation above as an equation for u, and put it the way below.

$u^2 + (v - 1)u + v^2 + v + 1 = 0$, which is a quadratic equation for u.   So?

If the equation has the solution, the solution has to be real.
That is, the equation needs to have a real root.
And since the equation above needs to have a real root, the discriminant D has to be ≥ 0.

That is, we need this: $D = (v - 1)^2 - 4{\cdot}1{\cdot}(v^2 + v + 1) \geq 0$.

Meanwhile,
$D = v^2 - 2v + 1 - 4v^2 - 4v - 4 = -3v^2 - 6v - 3 = -3(v^2 + 2v + 1) = -3(v + 1)^2$.

So unless $v = -1$, we get $D = -3(v + 1)^2 < 0$, that is, $D < 0$, which is not what we want.
What then do we need to have?

We need to have $v = \log y = -1$, since we need to have $D \geq 0$.

So by the definition for logs, we get $y = 10^{-1} = \frac{1}{10} = 0.1$.   How?

We know $\log y = \log_{10} y$, and the definition for logs is $A = b^x \Leftrightarrow x = \log_b A$.

So by the definition for logs, we can get $y = 10^{-1} \Leftrightarrow -1 = \log_{10} y = \log y = \log 10^{-1}$.

What then is the next?

We want to get the value of u so that we can get the value of x, since we set: $u = \log x$.

So next, going back to the equation $u^2 + (v - 1)u + v^2 + v + 1 = 0$, since $v = -1$, we get

$u^2 + (v - 1)u + v^2 + v + 1 = u^2 - 2u + 1 = (u - 1)^2 = 0$.

So we get $u = 1$, and thus, we get $\log x = 1 \Rightarrow x = 10$.

Therefore, $xy = 1$.

In short:

Setting $u = \log x$ and $v = \log y$, we get

$$(\log xy)^2 = (\log x + \log y)^2 = (\log x)(\log y) + \log x - \log y - 1$$
$$\Rightarrow (u + v)^2 = uv + u - v - 1 \Rightarrow u^2 + 2uv + v^2 = uv + u - v - 1$$
$$\Rightarrow u^2 + uv - u + v + v^2 + 1 = 0.$$

Taking the final equation above as an equation for u, we can get a quadratic equation as follows. $u^2 + (v - 1)u + v^2 + v + 1 = 0$.

Since the equaion above needs to have real roots, we need to have
$D = (v - 1)^2 - 4(v^2 + v + 1) \geq 0$.

Expanding the expression above, we get

$$D = v^2 - 2v + 1 - 4v^2 - 4v - 4 = -3v^2 - 6v - 3 = -3(v^2 + 2v + 1) = -3(v + 1)^2 \geq 0$$
$$\Rightarrow v = -1 \Rightarrow \log y = -1.$$

Thus, $y = 10^{-1}$.

Next, $v = -1 \Rightarrow u^2 + (v - 1)u + v^2 + v + 1 = u^2 - 2u + 1 = (u - 1)^2 = 0$.

So $u = 1$, and thus, $\log x = 1 \Rightarrow x = 10$. Therefore, $xy = 1$.

Suggestions or Solutions
To the Problem 1 in the Example 0

Assuming that ab, bc, and $ca > 0$, and that $(\log ac)(\log bc) + 1 = 0$ is an equation for c, find an additional condition on a and b so that the equation has real roots.

$(\log ac)(\log bc) + 1 = (\log a + \log c)(\log b + \log c) + 1$.

Setting $u = \log a$, $v = \log b$, and $w = \log c$, we get
$(u + w)(v + w) + 1 = w^2 + (u + v)w + uv + 1 = 0$.

Since the equation above needs to have real roots, the discriminant $D \geq 0$.
Taking thus, the discriminant D, and setting it to be ≥ 0, we get
$D = (u + v)^2 - 4(uv + 1) = u^2 + 2uv + v^2 - 4uv - 4 = u^2 - 2uv + v^2 - 4 = (u - v)^2 - 4 \geq 0$.

So we get $(u - v)^2 \geq 4 \Rightarrow u - v \leq -2$, or $u - v \geq 2$.

We have $u - v = \log a - \log b = \log\frac{a}{b}$. Thus, we get $\log\frac{a}{b} \leq -2$, or $\log\frac{a}{b} \geq 2$.

Therefore (by the definition for logs), we get $\frac{a}{b} \leq 10^{-2}$, or $\frac{a}{b} \geq 10^{2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what do we mean by an equation with real roots?

The equation is probably quadratic, so it is very likely that we work with its discriminant. How can we see though, it's a quadratic equation?

We can begin with expanding the left hand side of the equation given so that we can see better the unknown c.

Then, we get $(\log ac)(\log bc) + 1 = (\log a + \log c)(\log b + \log c) + 1 = 0$.

And next, since the equation above is an equation for c, taking a and b as constants, and taking $\log c$ as the unknown, we can take the equation as a quadratic equation for $\log c$.

Setting in fact, $u = \log a$, $v = \log b$, and $w = \log c$, we get a quadratic equation for w as follows. $(u + w)(v + w) + 1 = w^2 + (u + v)w + uv + 1 = 0$.

Since the equation above needs to have real roots, the discriminant $D \geq 0$. So we get

$D = (u + v)^2 - 4(uv + 1) = u^2 + 2uv + v^2 - 4uv - 4 = u^2 - 2uv + v^2 - 4 = (u - v)^2 - 4 \geq 0$.

Thus, we get $(u - v)^2 \geq 4 \Rightarrow u - v \leq -2$, or $u - v \geq 2$.

Now, since we have $u = \log a$, and $v = \log b$, we get $u - v = \log a - \log b = \log \frac{a}{b}$.

So we get $\log \frac{a}{b} \leq -2$, or $\log \frac{a}{b} \geq 2$. Therefore, we get $\frac{a}{b} \leq 10^{-2}$, or $\frac{a}{b} \geq 10^2$.

We can put it this way, too: $100a \leq b$, or $a \geq 100b$.

In short:

$(\log ac)(\log bc) + 1 = (\log a + \log c)(\log b + \log c) + 1$.

Setting $u = \log a$, $v = \log b$, and $w = \log c$, we get

$(u + w)(v + w) + 1 = w^2 + (u + v)w + uv + 1 = 0$.

Since the equation above needs to have real roots, the discriminant $D \geq 0$.
Taking thus, the discriminant D, and setting it to be ≥ 0, we get
$D = (u + v)^2 - 4(uv + 1) = u^2 + 2uv + v^2 - 4uv - 4 = u^2 - 2uv + v^2 - 4 = (u - v)^2 - 4 \geq 0$.

So we get $(u - v)^2 \geq 4 \Rightarrow u - v \leq -2$, or $u - v \geq 2$.

We have $u - v = \log a - \log b = \log \frac{a}{b}$. Thus, we get $\log \frac{a}{b} \leq -2$, or $\log \frac{a}{b} \geq 2$.

Therefore (by the definition for logs), we get $\frac{a}{b} \leq 10^{-2}$, or $\frac{a}{b} \geq 10^2$.

Suggestions or Solutions
To the Problem 0 in the Example 1

Assuming an equation for x, $(\log 2x)(\log 3x) = -a^2$ has two real roots, find the extent of a.

$(\log 2x)(\log 3x) = (\log 2 + \log x)(\log 3 + \log x)$

$= (\log x)^2 + (\log 2 + \log 3)(\log x) + (\log 2)(\log 3) = -a^2$.

Thus, setting $u = \log x$, we get $u^2 + (\log 6)u + (\log 2)(\log 3) + a^2 = 0$.

The equation above is a quadratic for u, and has two real roots, so the discriminant $D > 0$.

So we get $D = (\log 6)^2 - 4(\log 2)(\log 3) - 4a^2 > 0$.

Meanwhile, $\log 6 = \log 3 + \log 2 \Rightarrow (\log 6)^2 = (\log 3)^2 + 2(\log 3)(\log 2) + (\log 2)^2$.

Thus, $D = (\log 6)^2 - 4(\log 2)(\log 3) - 4a^2 = (\log 3)^2 - 2(\log 3)(\log 2) + (\log 2)^2 - 4a^2$

$= (\log 3 - \log 2)^2 - 4a^2 = (\log \frac{3}{2})^2 - 4a^2 = (\log \frac{3}{2} + 2a)(\log \frac{3}{2} - 2a) > 0$.

Therefore, $-\log \frac{3}{2} < 2a < \log \frac{3}{2} \Rightarrow -\frac{1}{2}\log \frac{3}{2} < a < \frac{1}{2}\log \frac{3}{2} \Rightarrow \log \sqrt{\frac{2}{3}} < a < \log \sqrt{\frac{3}{2}}$.

If not quite sure of the idea behind the processes above, follow the steps below.

The equation given is similar in structure to the one in the previous example.
So we can expect that the equation is quadratic with respect to $\log x$.

So let’s begin with expanding the left hand side so that we can expose the equation.

$(\log 2x)(\log 3x) = (\log 2 + \log x)(\log 3 + \log x)$

$= (\log x)^2 + (\log 2 + \log 3)(\log x) + (\log 2)(\log 3) = -a^2$.

Next, setting $\boldsymbol{u} = \mathbf{log}\ \boldsymbol{x}$, we get $\boldsymbol{u}^2 + (\mathbf{log}\ 6)\boldsymbol{u} + (\mathbf{log}\ 2)(\mathbf{log}\ 3) + \boldsymbol{a}^2 = 0$.

Now, the equation is a quadratic for $\boldsymbol{u}$, and has to have two real roots, so the discriminant $\boldsymbol{D}$ is not $\geq \mathbf{0}$ but $> \mathbf{0}$.

So we get $\boldsymbol{D} = (\log 6)^2 - 4\cdot 1\cdot\{(\log 2)(\log 3) + \boldsymbol{a}^2\} = (\log 6)^2 - 4(\log 2)(\log 3) - 4\boldsymbol{a}^2 > 0$.

Now, we could fill in the answer sheet the way as follows.

$(\log 6)^2 - 4(\log 2)(\log 3) - 4a^2 > 0 \Rightarrow 4a^2 < (\log 6)^2 - 4(\log 2)(\log 3)$

$$\Rightarrow a^2 < \frac{(\log 6)^2 - 4(\log 2)(\log 3)}{4}$$

$$\Rightarrow -\frac{\sqrt{(\log 6)^2 - 4(\log 2)(\log 3)}}{2} < a < \frac{\sqrt{(\log 6)^2 - 4(\log 2)(\log 3)}}{2}.$$

What if however, the test is multiple-choice, and the last expression above doesn't show up in the list of choices, but the list has $-\log\sqrt{\frac{2}{3}} < a < \log\sqrt{\frac{2}{3}}$, instead?

Notice that $\log 6 = \log(3\cdot 2) = \log 3 + \log 2$.

So we get $(\log 6)^2 = (\log 3 + \log 2)^2 = (\log 3)^2 + 2(\log 3)(\log 2) + (\log 2)^2$.

Thus, $\boldsymbol{D} = (\log 6)^2 - 4(\log 2)(\log 3) - 4a^2 = (\log 3)^2 - 2(\log 3)(\log 2) + (\log 2)^2 - 4a^2$

$$= (\log 3 - \log 2)^2 - 4a^2 = (\log\tfrac{3}{2})^2 - 4a^2 = (\log\tfrac{3}{2} + 2a)(\log\tfrac{3}{2} - 2a) > 0.$$

So we get $-\log\frac{3}{2} < 2a < \log\frac{3}{2} \Rightarrow -\frac{1}{2}\log\frac{3}{2} < a < \frac{1}{2}\log\frac{3}{2} \Rightarrow -\log(\frac{3}{2})^{\frac{1}{2}} < a < \frac{1}{2}\log(\frac{3}{2})^{\frac{1}{2}}$.

Therefore, we can see this: $-\log\sqrt{\frac{3}{2}} < a < \log\sqrt{\frac{3}{2}}$.

Besides, we can put it this way, too: $\log\sqrt{\frac{2}{3}} < a < \log\sqrt{\frac{3}{2}}$.

That's because $-\log\sqrt{\frac{3}{2}} = \log(\frac{3}{2})^{-\frac{1}{2}} = \log(\frac{2}{3})^{\frac{1}{2}} = \log\sqrt{\frac{2}{3}}$.

In short:

$(\log 2x)(\log 3x) = (\log 2 + \log x)(\log 3 + \log x)$

$= (\log x)^2 + (\log 2 + \log 3)(\log x) + (\log 2)(\log 3) = -a^2$.

Thus, setting $u = \log x$, we get $u^2 + (\log 6)u + (\log 2)(\log 3) + a^2 = 0$.

The equation above is a quadratic for u, and has two real roots, so the discriminant $D > 0$.

So we get $D = (\log 6)^2 - 4(\log 2)(\log 3) - 4a^2 > 0$.

Meanwhile, $\log 6 = \log 3 + \log 2 \Rightarrow (\log 6)^2 = (\log 3)^2 + 2(\log 3)(\log 2) + (\log 2)^2$.

Thus, $D = (\log 6)^2 - 4(\log 2)(\log 3) - 4a^2 = (\log 3)^2 - 2(\log 3)(\log 2) + (\log 2)^2 - 4a^2$

$= (\log 3 - \log 2)^2 - 4a^2 = (\log\frac{3}{2})^2 - 4a^2 = (\log\frac{3}{2} + 2a)(\log\frac{3}{2} - 2a) > 0$.

Therefore, we get

$$-\log\frac{3}{2} < 2a < \log\frac{3}{2} \Rightarrow -\frac{1}{2}\log\frac{3}{2} < a < \frac{1}{2}\log\frac{3}{2} \Rightarrow \log\sqrt{\frac{2}{3}} < a < \log\sqrt{\frac{3}{2}}.$$

Suggestions or Solutions
To the Problem 1 in the Example 1

Assuming an equation for x, $(\log 2x)(\log 3x) = -a^2$ has two real roots, find the value of the product of the roots.

Setting $\boldsymbol{u = \log x}$, we get $\boldsymbol{u^2 + (\log 6)u + (\log 2)(\log 3) + a^2 = 0}$.

Suppose that $\boldsymbol{s}$ and $\boldsymbol{t}$ are the two roots for the equation above.

Then, $\boldsymbol{(u - s)(u + t) = 0 \Rightarrow u^2 - (s + t)u + st = 0}$. Thus, $\boldsymbol{s + t = -\log 6 = \log 6^{-1}}$.

We have $\boldsymbol{u = \log x}$, too.
So we get $\boldsymbol{\log x = s}$ or $\boldsymbol{t \Rightarrow x = 10^s}$ or $\boldsymbol{10^t}$.

Therefore, the product is $\boldsymbol{10^s 10^t = 10^{s+t} = 10^{\log 6^{-1}} = 6^{-1} = \frac{1}{6}}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Setting $\boldsymbol{u = \log x}$, we get $\boldsymbol{u^2 + (\log 6)u + (\log 2)(\log 3) + a^2 = 0}$.

We don't even know however, the values of the roots for the equation above.

We can't get the actual values of the roots unless the value of $\boldsymbol{a}$ is chosen. In fact, the actual values of the true roots are the values of $\boldsymbol{x}$ that can satisfy the original equation.

Thus, it is even more impossible to get the actual values. So not even knowing those values, how can we get the value of the product of the true roots?

Examining the coefficients however, we could get the product or sum of the roots.

For this equation, we can get the value of the product without knowing the values of the true roots.

Suppose that s and t are the two roots for the equation for u above.

Then, $u = s$ or t. Thus, we get $(u - s)(u + t) = 0 \Rightarrow u^2 - (s + t)u + st = 0$.

So we can see that $-(s + t) = \log 6$, and that $st = (\log 2)(\log 3) + a^2$.

Thus, we get $s + t = -\log 6 = \log 6^{-1}$. What then is the next?

We have $u = \log x$, too. So we get $\log x = s$ or t. Thus, we get $x = 10^s$ or 10^t.

So we get $10^s 10^t = 10^{s+t}$ as the product of the two true roots.

We have $s + t = \log 6^{-1}$, too. So the product is $10^{s+t} = 10^{\log 6^{-1}} = 6^{-1} = \frac{1}{6}$. How?

Assuming $y = b^{\log_b A}$, and applying $\log_b$ to both sides of the equality, we get

$$\log_b y = \log_b b^{\log_b A} = (\log_b A)(\log_b b) = \log_b A.$$

Thus, we get $\log_b y = \log_b A$, so we get $y = A$.

Now, taking $b = 10$, we get

$10^{\log 6^{-1}} = 6^{-1}$, since $\log 6^{-1}$ is a common log, and thus, is the same as $\log_{10} 6^{-1}$.

Suggestions or Solutions
To the Problem in the Example 2

Solve the following equation for x.

$$\left(\frac{x}{10^{100}}\right)^{u} = \frac{\log 10^{x}}{x} \text{ where } u = (\log x^{v} - \log x^{7} + 10)^{3} \text{ where } v = \log x.$$

$\left(\frac{x}{10^{100}}\right)^{u} = \frac{\log 10^{x}}{x} = \frac{x\log 10}{x} = \log 10 = 1 \Rightarrow \left(\frac{x}{10^{100}}\right)^{u} = 1 \Rightarrow u = 0$ or $\frac{x}{10^{100}} = 1$.

So first, $\frac{x}{10^{100}} = 1 \Rightarrow x = 10^{100}$.

Next, $u = 0 \Rightarrow (\log x^{v} - \log x^{7} + 10)^{3} = 0 \Rightarrow \log x^{v} - \log x^{7} + 10 = 0$.

So $v = \log x \Rightarrow \log x^{v} - \log x^{7} + 10 = v \log x - 7 \log x + 10 = (\log x)^{2} - 7 \log x + 10 = 0$.

Thus, $(\log x - 5)(\log x - 2) = 0 \Rightarrow \log x = 5$ or $2 \Rightarrow x = 10^{5}$ or 10^{2}.

Therefore, $x = 10^{100}$, 10^{5}, or 10^{2}.

If not quite sure of the idea behind the processes above, follow the steps below.

This example looks awfully complicated, doesn't it?

Quite often however, the solution to such a problem is pretty simple.

Examining the equation, we can notice that the right hand side simply reduces to 1.

We can have $\frac{\log 10^{x}}{x} = \frac{x\log 10}{x} = \log 10 = 1$. Thus, we can see that $\left(\frac{x}{10^{100}}\right)^{u} = 1$.

Thus, we simply get $\boldsymbol{u} = \mathbf{0}$, or $\frac{\boldsymbol{x}}{\mathbf{10^{100}}} = \mathbf{1}$. So we get $\boldsymbol{x} = \mathbf{10^{100}}$, for now.

Let's next, get this one done: $\boldsymbol{u} = \mathbf{0}$.

We have $\boldsymbol{u} = \mathbf{(\log} \boldsymbol{x^v} \mathbf{- \log} \boldsymbol{x}^{\mathbf{7}} \mathbf{+ 10)^3}$.

Since $\boldsymbol{u} = \mathbf{0}$, we simply get $\mathbf{\log} \boldsymbol{x^v} \mathbf{- \log} \boldsymbol{x}^{\mathbf{7}} \mathbf{+ 10 = 0}$. We have $\boldsymbol{v} = \mathbf{\log} \boldsymbol{x}$, too. So?

We can get $\mathbf{\log} \boldsymbol{x^v} \mathbf{- \log} \boldsymbol{x}^{\mathbf{7}} \mathbf{+ 10} = \boldsymbol{v} \mathbf{\log} \boldsymbol{x} \mathbf{- 7 \log} \boldsymbol{x} \mathbf{+ 10 = (\log} \boldsymbol{x}\mathbf{)^2 - 7 \log} \boldsymbol{x} \mathbf{+ 10 = 0}$.

So we get $\mathbf{(\log} \boldsymbol{x} \mathbf{- 5)(\log} \boldsymbol{x} \mathbf{- 2) = 0} \Rightarrow \mathbf{\log} \boldsymbol{x} \mathbf{= 5}$ or $\mathbf{2} \Rightarrow \boldsymbol{x} = \mathbf{10^5}$ or $\mathbf{10^2}$.

Therefore, $\boldsymbol{x} = \mathbf{10^{100}}$, $\mathbf{10^5}$, or $\mathbf{10^2}$.

In short:

$$\left(\frac{\boldsymbol{x}}{\mathbf{10^{100}}}\right)^{\boldsymbol{u}} = \frac{\mathbf{\log 10^{\boldsymbol{x}}}}{\boldsymbol{x}} = \frac{\boldsymbol{x}\mathbf{\log 10}}{\boldsymbol{x}} = \mathbf{\log 10 = 1} \Rightarrow \left(\frac{\boldsymbol{x}}{\mathbf{10^{100}}}\right)^{\boldsymbol{u}} = \mathbf{1} \Rightarrow \boldsymbol{u} = \mathbf{0} \text{ or } \frac{\boldsymbol{x}}{\mathbf{10^{100}}} = \mathbf{1}.$$

So first, $\frac{\boldsymbol{x}}{\mathbf{10^{100}}} = \mathbf{1} \Rightarrow \boldsymbol{x} = \mathbf{10^{100}}$.

Next, $\boldsymbol{u} = \mathbf{0} \Rightarrow \mathbf{(\log} \boldsymbol{x^v} \mathbf{- \log} \boldsymbol{x}^{\mathbf{7}} \mathbf{+ 10)^3 = 0} \Rightarrow \mathbf{\log} \boldsymbol{x^v} \mathbf{- \log} \boldsymbol{x}^{\mathbf{7}} \mathbf{+ 10 = 0}$.

So $\boldsymbol{v} = \mathbf{\log} \boldsymbol{x} \Rightarrow \mathbf{\log} \boldsymbol{x^v} \mathbf{- \log} \boldsymbol{x}^{\mathbf{7}} \mathbf{+ 10} = \boldsymbol{v} \mathbf{\log} \boldsymbol{x} \mathbf{- 7 \log} \boldsymbol{x} \mathbf{+ 10 = (\log} \boldsymbol{x}\mathbf{)^2 - 7 \log} \boldsymbol{x} \mathbf{+ 10 = 0}$.

Thus, $\mathbf{(\log} \boldsymbol{x} \mathbf{- 5)(\log} \boldsymbol{x} \mathbf{- 2) = 0} \Rightarrow \mathbf{\log} \boldsymbol{x} \mathbf{= 5}$ or $\mathbf{2} \Rightarrow \boldsymbol{x} = \mathbf{10^5}$ or $\mathbf{10^2}$.

Therefore, $\boldsymbol{x} = \mathbf{10^{100}}$, $\mathbf{10^5}$, or $\mathbf{10^2}$.