Examples C on Logarithms

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0. Assuming $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d} > \mathbf{0}$, but $\neq \mathbf{1}$, and $\boldsymbol{b}^2 = \boldsymbol{ac}$, show $\dfrac{\log_a d}{\log_c d} = \dfrac{\log_a d - \log_b d}{\log_b d - \log_c d}$.

1. Solve $(\log_3 x)\sqrt{\log_x \sqrt{3x}} = -1$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming a, b, c, and $d > 0$, but $\neq 1$, and $b^2 = ac$, show $\dfrac{\log_a d}{\log_c d} = \dfrac{(\log_a d - \log_b d)}{(\log_b d - \log_c d)}$.

On the left hand side, we get $\dfrac{\log_a d}{\log_c d} = \dfrac{\frac{\log d}{\log a}}{\frac{\log d}{\log c}} = \dfrac{\log c}{\log a} = \log_a c$.

On the right hand side,

$$\log_a d - \log_b d = \tfrac{\log d}{\log a} - \tfrac{\log d}{\log b} = \log d\left(\tfrac{1}{\log a} - \tfrac{1}{\log b}\right) = \log d \,\tfrac{\log b - \log a}{(\log a)(\log b)}.$$

$$\log_b d - \log_c d = \tfrac{\log d}{\log b} - \tfrac{\log d}{\log c} = \log d\left(\tfrac{1}{\log b} - \tfrac{1}{\log c}\right) = \log d \,\tfrac{\log c - \log b}{(\log b)(\log c)}.$$

Thus, $\dfrac{\log_a d - \log_b d}{\log_b d - \log_c d} = \dfrac{\log d \frac{\log b - \log a}{(\log a)(\log b)}}{\log d \frac{\log c - \log b}{(\log b)(\log c)}} = \dfrac{\frac{\log b - \log a}{(\log a)(\log b)}}{\frac{\log c - \log b}{(\log b)(\log c)}} = \dfrac{\log b - \log a}{(\log a)(\log b)} \cdot \dfrac{(\log b)(\log c)}{\log c - \log b}$

$$= \frac{\log b - \log a}{\log a} \cdot \frac{\log c}{\log c - \log b} = \frac{\log c}{\log a} \cdot \frac{\log b - \log a}{\log c - \log b} = \log_a c \cdot \frac{\log b - \log a}{\log c - \log b}.$$

So we now, have $\dfrac{\log_a d}{\log_c d} = \log_a c$, and $\dfrac{\log_a d - \log_b d}{\log_b d - \log_c d} = \log_a c \cdot \dfrac{\log b - \log a}{\log c - \log b}$.

Thus, we need to have $\dfrac{\log b - \log a}{\log c - \log b} = 1$. And we have $b^2 = ac$. So we get

$\log b^2 = \log ac \Rightarrow 2 \log b = \log ac = \log a + \log c \Rightarrow 2 \log b = \log a + \log c$

$\Rightarrow \log b - \log a = \log c - \log b$.

Thus, we get $\dfrac{\log b - \log a}{\log c - \log b} = 1$.

So we now, have $\dfrac{\log_a d}{\log_c d} = \log_a c$, and also, $\dfrac{\log_a d - \log_b d}{\log_b d - \log_c d} = \log_a c$.

Therefore, $\dfrac{\log_a d}{\log_c d} = \dfrac{(\log_a d - \log_b d)}{(\log_b d - \log_c d)}$ where $b^2 = ac$.

If not quite sure of the idea behind the processes above, follow the steps below.

A log can be put in a fractional form.

For instance, we can modify $\log_x y$ to $\dfrac{\log y}{\log x}$. That is, we have $\log_x y = \dfrac{\log y}{\log x}$.

And we call the equality a log identity. So?

So we can use such an identity, together with other identities on logs, of course, when working with a log expression like those in the equality above.

Examining the equality given, we can see that all the logs have the same antilog, which is d. Thus, we can expect that d can be removed (actually, canceled out). It is like canceling out processes in fractional multiplications.

For instance, we can have $\dfrac{3}{4} \cdot \dfrac{4}{5} = \dfrac{3}{5}$, where 4s get canceled out.

So how are we going to get this problem done?

If we can modify both sides of the equality so that both sides get turned out to be the same expression, the equality is proven. And doing the modifications, we are going to use the identity above. So let’s now, begin with the left hand side.

$$\frac{\log_a d}{\log_c d} = \frac{\frac{\log d}{\log a}}{\frac{\log d}{\log c}} = \frac{\log d}{\log a} \cdot \frac{\log c}{\log d} = \frac{\log c}{\log a} = \log_a c.$$

Next, let's move on to the right hand side, and begin with the numerator.

$$\log_a d - \log_b d = \frac{\log d}{\log a} - \frac{\log d}{\log b} = \log d\left(\frac{1}{\log a} - \frac{1}{\log b}\right) = \log d \, \frac{\log b - \log a}{(\log a)(\log b)}.$$

And next, moving on to the denominator, we can get

$$\log_b d - \log_c d = \frac{\log d}{\log b} - \frac{\log d}{\log c} = \log d\left(\frac{1}{\log b} - \frac{1}{\log c}\right) = \log d \, \frac{\log c - \log b}{(\log b)(\log c)}.$$

Thus, we get

$$\frac{\log_a d - \log_b d}{\log_b d - \log_c d} = \frac{\log d \, \frac{\log b - \log a}{(\log a)(\log b)}}{\log d \, \frac{\log c - \log b}{(\log b)(\log c)}} = \frac{\frac{\log b - \log a}{(\log a)(\log b)}}{\frac{\log c - \log b}{(\log b)(\log c)}} = \frac{\log b - \log a}{(\log a)(\log b)} \cdot \frac{(\log b)(\log c)}{\log c - \log b}$$

$$= \frac{\log b - \log a}{\log a} \cdot \frac{\log c}{\log c - \log b} = \frac{\log c}{\log a} \cdot \frac{\log b - \log a}{\log c - \log b} = \log_a c \cdot \frac{\log b - \log a}{\log c - \log b}.$$

So we now, have $\frac{\log_a d}{\log_c d} = \log_a c$, and $\frac{\log_a d - \log_b d}{\log_b d - \log_c d} = \log_a c \cdot \frac{\log b - \log a}{\log c - \log b}$.

Thus, we get $\frac{\log_a d}{\log_c d} = \frac{(\log_a d - \log_b d)}{(\log_b d - \log_c d)} \Rightarrow \log_a c = \log_a c \cdot \frac{\log b - \log a}{\log c - \log b}$. So what?

We want to show this: $\frac{\log b - \log a}{\log c - \log b} = 1$. How though?

We can show this, instead: **log b – log a = log c – log b**.

We can't just show that of course. We need to have something to work with.

And we haven't used yet something that is given. What then is it?

It is the supposition that $\boldsymbol{b^2 = ac}$, and we want to take advantage of it.
How can we use it though?

We are now working with logs. So?

So taking advantage of the supposition, we can apply logs to it. Then, we get

$\mathbf{\log b^2 = \log ac = \log a + \log c \Rightarrow 2\log b = \log a + \log c}$.

Thus, we get $\mathbf{\log b = \log a - \log c - \log b \Rightarrow \log b - \log a = \log c - \log b}$.

In short:

On the left hand side, we get $\dfrac{\log_a d}{\log_c d} = \dfrac{\frac{\log d}{\log a}}{\frac{\log d}{\log c}} = \dfrac{\log c}{\log a} = \log_a c$.

On the right hand side,

$$\log_a d - \log_b d = \frac{\log d}{\log a} - \frac{\log d}{\log b} = \log d\left(\frac{1}{\log a} - \frac{1}{\log b}\right) = \log d\,\frac{\log b - \log a}{(\log a)(\log b)}.$$

$$\log_b d - \log_c d = \frac{\log d}{\log b} - \frac{\log d}{\log c} = \log d\left(\frac{1}{\log b} - \frac{1}{\log c}\right) = \log d\,\frac{\log c - \log b}{(\log b)(\log c)}.$$

Thus, $\dfrac{\log_a d - \log_b d}{\log_b d - \log_c d} = \dfrac{\log d \,\frac{\log b - \log a}{(\log a)(\log b)}}{\log d \,\frac{\log c - \log b}{(\log b)(\log c)}} = \dfrac{\frac{\log b - \log a}{(\log a)(\log b)}}{\frac{\log c - \log b}{(\log b)(\log c)}} = \dfrac{\log b - \log a}{(\log a)(\log b)} \cdot \dfrac{(\log b)(\log c)}{\log c - \log b}$

$= \dfrac{\log b - \log a}{\log a} \cdot \dfrac{\log c}{\log c - \log b} = \dfrac{\log c}{\log a} \cdot \dfrac{\log b - \log a}{\log c - \log b} = \log_a c \cdot \dfrac{\log b - \log a}{\log c - \log b}$.

So we now, have $\dfrac{\log_a d}{\log_c d} = \log_a c$, and $\dfrac{\log_a d - \log_b d}{\log_b d - \log_c d} = \log_a c \cdot \dfrac{\log b - \log a}{\log c - \log b}$.

Thus, we need to have $\dfrac{\log b - \log a}{\log c - \log b} = 1$. And we have $b^2 = ac$.

So we get $\log b^2 = \log ac \Rightarrow 2 \log b = \log ac = \log a + \log c \Rightarrow 2 \log b = \log a + \log c$

$\Rightarrow \log b - \log a = \log c - \log b$.

Thus, we get $\dfrac{\log b - \log a}{\log c - \log b} = 1$.

So we now, have $\dfrac{\log_a d}{\log_c d} = \log_a c$, and also, $\dfrac{\log_a d - \log_b d}{\log_b d - \log_c d} = \log_a c$.

Therefore, $\dfrac{\log_a d}{\log_c d} = \dfrac{(\log_a d - \log_b d)}{(\log_b d - \log_c d)}$ where $b^2 = ac$.

Suggestions or Solutions
To the Problem in the Example 1

Solve for x in the equation, $(\log_3 x)\sqrt{\log_x \sqrt{3x}} = -1$.

$$(\log_3 x)\sqrt{\log_x \sqrt{3x}} = -1 \Rightarrow \sqrt{\log_x \sqrt{3x}} = -\log_x 3 \Rightarrow \log_x \sqrt{3x} = (\log_x 3)^2$$

$$\Rightarrow \tfrac{1}{2}\log_x 3x = (\log_x 3)^2 \Rightarrow \log_x 3x = 2(\log_x 3)^2 \Rightarrow \log_x 3 + 1 = 2(\log_x 3)^2.$$

Setting $\log_x 3 = t$, we get $2t^2 - t - 1 = 0 \Rightarrow (2t+1)(t-1) = 0 \Rightarrow t = -\frac{1}{2}$ or 1.

$$t = -\tfrac{1}{2} \Rightarrow \log_x 3 = -\tfrac{1}{2} \Rightarrow 3 = x^{-\frac{1}{2}} \Rightarrow x^{-1} = 9 \Rightarrow x = 9^{-1}.$$

$$t = 1 \Rightarrow \log_x 3 = 1 \Rightarrow x = 3.$$

$$x = 3 \Rightarrow (\log_3 x)\sqrt{\log_x \sqrt{3x}} = (\log_3 3)\sqrt{\log_3 \sqrt{9}} = \sqrt{\log_3 \sqrt{9}} = \sqrt{\log_3 3} = 1 \neq -1.$$

Thus, $x = 3$ is not allowed.

$$x = 9^{-1} \Rightarrow (\log_3 x)\sqrt{\log_x \sqrt{3x}} = (\log_3 9^{-1})\sqrt{\log_{9^{-1}} \sqrt{3\cdot 9^{-1}}} = -2\sqrt{\log_{9^{-1}} \sqrt{3^{-1}}}$$

$$= -2\sqrt{\log_{3^{-2}} 3^{-\frac{1}{2}}} = -2\sqrt{\frac{-\frac{1}{2}}{-2}} = -2\sqrt{\frac{1}{4}} = -2\cdot\frac{1}{2} = -1.$$ Therefore, $x = 9^{-1}$.

If not quite sure of the idea behind the processes above, follow the steps below.

It looks like we have three unknowns in one equation.
They are the same though, of course, and x is the unknown.
In the equation, the same unknown x is in three different places.

It is inside a log, that is, it's an antilog, and is inside a square root sign, and also, it is used as a base in a log, too. So how are we going to do with it?

Basically, solving an equation, we are isolating the unknown so that we can eventually get a form as follows: $\boldsymbol{x}$ = a number.   How then can we get it isolated?

We can remove the square root sign.   How though?

First, we can isolate the square root part by moving $\mathbf{log_3}\ \boldsymbol{x}$ to the right hand side.   How?

We can divide both sides by $\mathbf{log_3}\ \boldsymbol{x}$ respectively.
So doing the divisions, and doing some more algebra, we can get:

$$\sqrt{\log_x \sqrt{3x}} = \frac{-1}{\log_3 x} = \frac{-1}{\frac{\log x}{\log 3}} = -\frac{\log 3}{\log x} = -\log_x 3 \Rightarrow \sqrt{\log_x \sqrt{3x}} = -\log_x 3.$$

Next, squaring both sides, we can remove the outer square root sign. Then, we get

$$\log_x \sqrt{3x} = (\log_x 3)^2.$$

Then, negative cases can get involved due to squaring, so we don't want to forget to check to see if all the solutions are appropriate. Now, moving on to the next step, we get

$$\log_x \sqrt{3x} = \tfrac{1}{2}\log_x 3x = (\log_x 3)^2 \Rightarrow \log_x 3x = 2(\log_x 3)^2.$$ What then?

We know: $\mathbf{log}_x\ \mathbf{3}x = \mathbf{log}_x\ \mathbf{3} + \mathbf{log}_x\ x = \mathbf{log}_x\ \mathbf{3} + \mathbf{1}$.   So we get: $\mathbf{log}_x\ \mathbf{3} + \mathbf{1} = \mathbf{2}(\mathbf{log}_x\ \mathbf{3})^2$.

What equation then do we get? Isn't it getting more complicated, is it?

We just get a quadratic equation, which is usually easy to solve. What quadratic equation though?

It is a quadratic equation for $\log_x 3$, and not just x. How then can we get it solved?

We can set, for instance, $u = \log_x 3$.
Then, we can get a quadratic equation for u.
So we solve that quadratic equation first. And then, we can solve $u = \log_x 3$ for x.

So setting now, $u = \log_x 3$, and solving the equation, we get

$$\log_x 3 + 1 = 2(\log_x 3)^2 \Rightarrow u + 1 = 2u^2 \Rightarrow 2u^2 - u - 1 = 0 \Rightarrow (2u + 1)(u - 1) = 0$$

$\Rightarrow u = -\frac{1}{2}$ or 1. Then, we get two equations.

One is $-\frac{1}{2} = \log_x 3$. And the other is $1 = \log_x 3$.

So first, by the definition for logs, we get $\log_x 3 = -\frac{1}{2} \Rightarrow 3 = x^{-\frac{1}{2}} \Rightarrow x^{-1} = 9 \Rightarrow x = 9^{-1}$.

And next, by the definition again, we get $\log_x 3 = 1 \Rightarrow x = 3$.

That's not it though. The quadratic equation we just have solved is not really the original equation given. So we want to see if the two roots are OK.

So assuming first, $x = 3$, we get

$$x = 3 \Rightarrow (\log_3 x)\sqrt{\log_x \sqrt{3x}} = (\log_3 3)\sqrt{\log_3 \sqrt{9}} = \sqrt{\log_3 \sqrt{9}} = \sqrt{\log_3 3} = 1 \neq -1.$$

Thus, $x = 3$ is not allowed.

And next, $x = 9^{-1} \Rightarrow (\log_3 x)\sqrt{\log_x \sqrt{3x}} = (\log_3 9^{-1})\sqrt{\log_{9^{-1}} \sqrt{3 \cdot 9^{-1}}} = -2\sqrt{\log_{9^{-1}} \sqrt{3^{-1}}}$

$$= -2\sqrt{\log_{3^{-2}} 3^{-\frac{1}{2}}} = -2\sqrt{\frac{-\frac{1}{2}}{-2}} = -2\sqrt{\frac{1}{4}} = -2 \cdot \frac{1}{2} = -1.$$

Therefore, $x = 9^{-1}$ is the solution.

In short:

$$(\log_3 x)\sqrt{\log_x \sqrt{3x}} = -1 \Rightarrow \sqrt{\log_x \sqrt{3x}} = -\log_x 3 \Rightarrow \log_x \sqrt{3x} = (\log_x 3)^2$$

$$\Rightarrow \tfrac{1}{2}\log_x 3x = (\log_x 3)^2 \Rightarrow \log_x 3x = 2(\log_x 3)^2 \Rightarrow \log_x 3 + 1 = 2(\log_x 3)^2.$$

Setting now, $\log_x 3 = t$, we can get $2t^2 - t - 1 = 0 \Rightarrow (2t+1)(t-1) = 0 \Rightarrow t = -\tfrac{1}{2}$ or 1.

$$t = -\tfrac{1}{2} \Rightarrow \log_x 3 = -\tfrac{1}{2} \Rightarrow 3 = x^{-\frac{1}{2}} \Rightarrow x^{-1} = 9 \Rightarrow x = 9^{-1}.$$

$$t = 1 \Rightarrow \log_x 3 = 1 \Rightarrow x = 3.$$

Now, we can check to see if the values are OK using this: $(\log_3 x)\sqrt{\log_x \sqrt{3x}} = -1$.

$$x = 3 \Rightarrow (\log_3 x)\sqrt{\log_x \sqrt{3x}} = (\log_3 3)\sqrt{\log_3 \sqrt{9}} = \sqrt{\log_3 \sqrt{9}} = \sqrt{\log_3 3} = 1 \neq -1.$$

Thus, $x = 3$ is not allowed.

$$x = 9^{-1} \Rightarrow (\log_3 x)\sqrt{\log_x \sqrt{3x}} = (\log_3 9^{-1})\sqrt{\log_{9^{-1}} \sqrt{3 \cdot 9^{-1}}} = -2\sqrt{\log_{9^{-1}} \sqrt{3^{-1}}}$$

$$= -2\sqrt{\log_{3^{-2}} 3^{-\frac{1}{2}}} = -2\sqrt{\frac{-\frac{1}{2}}{-2}} = -2\sqrt{\frac{1}{4}} = -2 \cdot \frac{1}{2} = -1.$$

Therefore, $x = 9^{-1}$ is the solution.