Examples D on Logarithms

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0. Using $\log 4.6065 = 0.6634$, and $\log 1.908 = 0.2805$, evaluate $1.908^{-1.2}$.

1. Assuming u is the index (or characteristic) of $\log x$, and v is that of $\log \frac{10}{x}$, find x that maximizes $u^2 - 2v^2$, and find the maximum value.

Suggestions or Solutions
To the Problem in the Example 0

Using log 4.6065 = 0.6634 and log 1.908 = 0.2805, evaluate $1.908^{-1.2}$.

log $1.908^{-1.2}$ = -1.2 log 1.908. And we have **log 1.908 = 0.2805**.

So we get **-1.2 log 1.908 = -1.2(0.2805) = -0.3366 = -1 + 1 - 0.3366 = -1 + 0.6634**.

And we have **log 4.6065 = 0.6634**.

So the value of **$1.908^{-1.2}$** is **0.46065**.

If not quite sure *of the idea behind the processes above, follow the steps below.*

In this problem, we are given a means to work with. What then is the means?

It is a set of two logs and their values.

One is **log 4.6065 = 0.6634**, and the other is **log 1.908 = 0.2805**.

So what do we do with those?

Since we need to work with log-values, we may want to take a log of the value to be evaluated, and the log is a common log, of course. And then, we apply the means to it.

So to begin with, taking the common of **$1.908^{-1.2}$**, we get **log $1.908^{-1.2}$ = -1.2 log 1.908**.

And we have **log 1.908 = 0.2805**.

So we get **-1.2 log 1.908 = -1.2(0.2805) = -0.3366**. What do we do with it then?

The log above is a common log, so it can be put in terms of the characteristic and the mantissa. And we know $\mathbf{0 \le mantissa < 1}$.

So we can get $\mathbf{\log 1.908^{-1.2} = -0.3366 = -1 + 1 - 0.3366 = -1 + 0.6634}$.

Now, we have another piece of the means, which is $\mathbf{\log 4.6065 = 0.6634}$, which is the same as the mantissa above. So what are we going to do with it?

Suppose now, that $\log \boldsymbol{A = c + m}$, where $\boldsymbol{A > 0}$, $\boldsymbol{c}$ is an integer called the characteristic, and $\boldsymbol{m}$ is the mantissa.

Then first, the highest place value in $\boldsymbol{A}$ is $\mathbf{10}^{c}$.
In particular, if $\boldsymbol{c < 0}$, the first nonzero appears in the c^{th} digit below the decimal point.

Next, we have another fact as follows.

• If we get the same mantissas after taking common logs of two numbers, the two numbers have the same sequence of digits.

So what are we going to do with the fact?

We have two common logs, which are $\mathbf{\log 1.908^{-1.2}}$ and $\mathbf{\log 4.6065}$, both of which share the same mantissa, which is $\mathbf{0.6634}$. So?

So $\mathbf{1.908^{-1.2}}$ has the same sequence of digits as $\mathbf{4.6065}$ has. Thus, the sequence of digits in the number $\mathbf{1.908^{-1.2}}$ is $\{\mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{6}, \mathbf{5}\}$.

Next, the characteristic of $\mathbf{\log 1.908^{-1.2}}$ is -1, so the first nonzero appears in the first digit below the decimal point.

Therefore, the value of $\mathbf{1.908^{-1.2}}$ is $\mathbf{0.46065}$.

In short:

log $1.908^{-1.2}$ = -1.2 log 1.908.   And we have **log 1.908 = 0.2805**.

So we get **-1.2 log 1.908 = -1.2(0.2805) = -0.3366 = -1 + 1 - 0.3366 = -1 + 0.6634**.

And we have **log 4.6065 = 0.6634**.

So the value of **$1.908^{-1.2}$** is **0.46065**.

Suggestions or Solutions
To the Problem in the Example 1

Assuming u is the index (or characteristic) of log x, and v is that of $\log\frac{10}{x}$, find x that maximizes $u^2 - 2v^2$, and find the maximum value.

Suppose $\log x = u + m$, where $0 \le m < 1$.

Then, we get $\log\frac{10}{x} = \log 10 - \log x = 1 - u - m \Rightarrow \log\frac{10}{x} = 1 - u - m$.

If $m = 0$, we get $\log\frac{10}{x} = 1 - u - m = 1 - u + 0$. So the mantissa in this case is 0.

If $0 < m < 1$, we get $\log\frac{10}{x} = 1 - u - m = 1 - u - 1 + 1 - m = -u + 1 - m$. So in this case the mantissa is $1 - m$.

And we know v is the index of $\log\frac{10}{x}$.

So beginning with the case where $m = 0$, and $v = 1 - u$, we get

$$u^2 - 2v^2 = u^2 - 2(1 - u)^2 = u^2 - 2(1 - 2u + u^2) = -u^2 + 4u - 2 = -(u^2 - 4u + 2)$$
$$= -(u^2 - 4u + 4 - 4 + 2) = -(u^2 - 4u + 4 - 2) = -(u - 2)^2 + 2 \le 2.$$

So if $m = 0$, when $u = 2$, $(u^2 - 2v^2)$ gets 2 as its maximum.
And we know $\log x = u + m$, where $0 \le m < 1$.
So when $u = 2$, we get $\log x = 2 + 0 = 2$, and thus, we get $x = 100$.

And next, moving on to the case where $0 < m < 1$, $v = -u$, we get

$$u^2 - 2v^2 = u^2 - 2u^2 = -u^2 \le 0.$$

So if $0 < m < 1$, when $u = 0$, $(u^2 - 2v^2)$ gets 0 as its maximum.
So when $u = 0$, we get $\log x = 0 + m = m$, and thus, we get $x = 10^m$ where $0 < m < 1$.
So when $u = 0$, we get $1 < x < 10$.

Putting thus, threads together, we can say that when $x = 100$, $u^2 - 2v^2$ gets its maximum, which is 2.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, we can put $\log_b A$ the way below.

$\log_b A = n + m$, where n is an integer called the index, $0 \le m < 1$, and m is called the mantissa.

And in particular, if the base $b = 10$, the log is called a common log, and n is often called the characteristic. Then, the highest place value in A is 10^n.

And we know the fact that u is the characteristic of $\log x$.

So we can put $\log x$ this way: $\log x = u + m$, where $0 \le m < 1$. What then about $\log \frac{10}{x}$?

We can get $\log \frac{10}{x} = \log 10 - \log x = 1 - u - m \Rightarrow \log \frac{10}{x} = 1 - u - m$.

Note that m is the mantissa of $\log x$, and not that of $\log \frac{10}{x}$. So?

So we need to determine the mantissa of $\log \frac{10}{x}$. And we have two cases to consider.

One is $m = 0$, and the other is $0 < m < 1$.

If $m = 0$, we get $\log \frac{10}{x} = 1 - u - m = 1 - u + 0$.

So the mantissa of $\log \frac{10}{x}$ is simply 0. And of course, the index is $1 - u$.

If however, $0 < m < 1$, we get $\log \frac{10}{x} = 1 - u - m = 1 - u - 1 + 1 - m$.

So we get $\log \frac{10}{x} = -u + (1 - m)$, and $0 < 1 - m < 1$.

So the mantissa of $\log \frac{10}{x}$ is $(1 - m)$. And of course, the index is $-u$.

The problem says though, the index of $\log \frac{10}{x}$ is v. So?

So we have two cases for v as follows.

- If $m = 0$, the index of $\log \frac{10}{x}$ is $1 - u$, so in this case, we have $v = 1 - u$.
- And next, if $0 < m < 1$, the index of $\log \frac{10}{x}$ is $-u$, so in this case, we have $v = -u$.

So in sum, we have two cases to consider doing this problem.
One is the case where $m = 0$, and $v = 1 - u$.
And the other is the case where $0 < m < 1$ and $v = -u$.

Let's now find x that maximizes $u^2 - 2v^2$, and find the maximum value.

Beginning with the case where $m = 0$, and $v = 1 - u$, we get

$$u^2 - 2v^2 = u^2 - 2(1 - u)^2 = u^2 - 2(1 - 2u + u^2) = -u^2 + 4u - 2 = -(u^2 - 4u + 2)$$
$$= -(u^2 - 4u + 4 - 4 + 2) = -(u^2 - 4u + 4 - 2) = -(u - 2)^2 + 2 \leq 2.$$

So in this case, 2 is the maximum of $u^2 - 2v^2$.
And when it gets its maximum, we get $u = 2$.

And we know $\log x = u + m$, where $0 \leq m < 1$.

So at the maximum, we get $\log x = u + m = 2 + 0 = 2$, and therefore, $x = 100$.

And next, moving on to the case where $0 < m < 1$ and $v = -u$, we get

$$u^2 - 2v^2 = u^2 - 2u^2 = -u^2 \leq 0.$$

Thus, in this case, when $u = 0$, $(u^2 - 2v^2)$ gets 0 as its maximum.

Then, when $u = 0$, we get $\log x = u + m = m$.

So by the definition for logs, we get $x = 10^m$, where $0 < m < 1$, and thus, $1 < x < 10$.

Thus, putting threads together, when $x = 100$, $u^2 - 2v^2$ gets 2, which is the maximum.

In short:

Suppose $\log x = u + m$, where $0 \le m < 1$.

Then, we get $\log\frac{10}{x} = \log 10 - \log x = 1 - u - m \Rightarrow \log\frac{10}{x} = 1 - u - m$.

If $m = 0$, we get $\log\frac{10}{x} = 1 - u - m = 1 - u + 0$. So the mantissa in this case is 0.

If $0 < m < 1$, we get $\log\frac{10}{x} = 1 - u - m = 1 - u - 1 + 1 - m = -u + 1 - m$. So in this case the mantissa is $1 - m$.

And we know v is the index of $\log\frac{10}{x}$.

So beginning with the case where $m = 0$, and $v = 1 - u$, we get

$$u^2 - 2v^2 = u^2 - 2(1 - u)^2 = u^2 - 2(1 - 2u + u^2) = -u^2 + 4u - 2 = -(u^2 - 4u + 2)$$

$$= -(u^2 - 4u + 4 - 4 + 2) = -(u^2 - 4u + 4 - 2) = -(u - 2)^2 + 2 \le 2.$$

So if $m = 0$, when $u = 2$, $(u^2 - 2v^2)$ gets 2 as its maximum.
And we know $\log x = u + m$, where $0 \le m < 1$.
So when $u = 2$, we get $\log x = 2 + 0 = 2$, and thus, we get $x = 100$.

And next, moving on to the case where $0 < m < 1$, $v = -u$, we get

$$u^2 - 2v^2 = u^2 - 2u^2 = -u^2 \le 0.$$

So if $0 < m < 1$, when $u = 0$, $(u^2 - 2v^2)$ gets 0 as its maximum.
And we know $\log x = u + m$, where $0 \le m < 1$.
So when $u = 0$, we get $\log x = 0 + m = m$, and thus, we get $x = 10^m$ where $0 < m < 1$.
So when $u = 0$, we get $1 < x < 10$.

Putting thus, threads together, we can say that when $x = 100$, $u^2 - 2v^2$ gets its maximum, which is 2.