Examples 7 in Ratios and Rates

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A clock usually has three hands rotating about their axes. In such a clock, we can easily see the hour hand and the minute hand making angles as they rotate.

The angle between the two hands depends on the time the clock indicates. In each example, we'll find the time it takes for the hands to make a particular angle called the target angle from the current time given, and will find the time in hours, minutes, and seconds. Also, we'll find the first moment of time when the target angle happens.

We'll specify the moment of time in such a manner as 'hh:mm:ss' using the 24-hour system where 'hh' stands for the hour, 'mm' for the minute, and 'ss' for the second. For instance, if the target angle happens at 9 o'clock 17 minutes 32 + 5/11 seconds in the morning, the answer should be in such a form as follows. 09:17:32 + 5/11.

0. Find the first moment of time when 180^o happens, and the time it takes for the hour hand and the minute hand to make the target angle 180^o if the current time is as follows.

0.0. midnight (00:00:00).

0.1. 1 o'clock (01:00:00)

0.2. 14 o'clock (14:00:00)

0.3. 7 o'clock (07:00:00)

0.4. 20 o'clock (20:00:00)

1. How many times do the two hands make 180^o in a day?

2. Find a formula if any. For instance, we may want to find a formula for each hour.

Suggestions or Solutions
To The Problem 0.0 In Examples 7

0.0. If the current time is midnight (00:00:00), find the first moment of time when the hands make a target angle 180^o, and the time it takes for the hands to make the target angle.

How do the hands make the angle from each other?

Each hand makes its own angle.
We may want to see how the hands make their angles.
That’s because the target angle is from the two angles.

How then the two angles form the target angle?

We have in fact, three angles to work with.
What three angles?

They are the target angle, the angle made by the hour hand, and the angle made by the minute hand.
So to begin with, we may want to name those three angles.

Suppose H is the angle made by the hour hand, M is the angle made by the minute hand, and d is the target angle, which is the angle between the two hands.

And it takes time to make angles.
So let t be the time it takes to make angles. Of course, all the three angles are made at the same time

As the time flows, the angles H, M, and d change.
The two hands make d from each other.　How?

The angle between the two is the angle d.
So the difference between H and M is d.
Now, how then does each of H, M, and d get made?

If H and M get made, d gets made, since the difference between the two is d. So we want to go after H and M.

The hands are moving at the same time, but M is moving faster, and H is slower. Thus, they are moving at different rates.

What rate?
It's a rate of change in angle. What rate of change in angle?

It is a time rate of change in angle. So we need to find a time rate of change in angle for each hand.

A time rate of change in angle is a speed, so it indicates how fast or slow an angle changes, how fast or slow an angle gets bigger or smaller.
The hands are moving at their own speeds. One is faster and the other is slower. How fast and how slow?
So we need a speed of each hand, a time rate of change in angle for each hand.
Let's now get the speed of the hour hand first.

It takes 12 hours for the hour hand to make a complete turn around the clock, so it covers 360^o in 12 hours.
Therefore, the time rate of change in angle is as follows.

$$\frac{360^o}{12 \text{ hours}} = \frac{30^o}{1 \text{ hour}} = 30^o \text{ / hour}$$

Let's name the rate to be h. Then, $h = 30^o/\text{hr}$.

What then about the minute hand?

It takes 1 hour for the minute hand to sweep though the plate in a clock and cover 360^o. So naming the rate to be m,

we get m = $\dfrac{360^o}{1\ \text{hour}}$ = 360^o/hr.

The target angle needs to happen within an hour.
So it will happen in some minutes. Thus, putting the rates in minutes, we get

$$h = \frac{30^o}{1\ \text{hour}} = \frac{30^o}{60\ \text{minutes}} = \frac{0.5^o}{1\ \text{minute}} = 0.5^o/\text{min}$$

$$m = \frac{360^o}{1\ \text{hour}} = \frac{360^o}{60\ \text{minutes}} = \frac{6^o}{1\ \text{minute}} = 6^o/\text{min}$$

Now, H is the angle made by the hour hand, and M is the angle made by the minute hand.
So after the amount of time t, we get H = ht and M = mt.
What then about the difference d?

The minute hand is faster than the hour hand, and the minute hand moves ahead of the hour hand. So M is bigger than H when the target angle happens.

Thus, we get d = M – H = mt – ht = (m – h)t.

And we have h = 0.5^o/min, and m = 6^o/min.

Thus, we get (m – h)t = 5.5t = d, which will be the target angle 180^o. So we want d to be 180^o.

Thus, we get 5.5t = 180 $\Rightarrow$ t = $\frac{180}{5.5}$ minutes.

So after $\frac{180}{5.5}$ minutes, the hands make 180^o from each other.

We need to put the moment of time in hour, minute, and second. So putting the time that way, we can get first,

t = $\frac{180}{5.5} = \frac{1800}{55} = \frac{360}{11}$ = 32 + $\frac{8}{11}$ minutes.

We know 1 minute = 60 seconds.

So next, we get $\frac{8}{11}$ x 60 = $\frac{480}{11} = 43 + \frac{7}{11}$ seconds.

Thus, the hands make 180^o at 00 : 32 : 43 + 7/11, which means, 180^o happens at 0 hour, 32 minutes, and 43 and 7/11 seconds.

When we work with things that have to do with changes, we probably need to work with rates.

So where there is a change, there is a rate.   What rate?

A rate of change.   What rate of change?

We often talk about a time rate of change in distance, which is usually called a speed, how fast or slow a distance gets covered. Usually thus, a speed is a time rate of change in distance.

Of course, we can talk about other rate of change such as an altitude rate of change in temperature.

For instance, a temperature drops 0.5^{o} C if the altitude increases in the amount of 100 meters. In other words, the temperature falls 0.5^{o} C per 100 meters in altitude.
Then, we can say that the rate is -5^{o} C/km, which is an altitude rate of change in temperature.

Suggestions or Solutions
To The Problem 0.1 In Examples 7

0.1. If the current time is 1 o'clock (01:00:00), find the first moment of time when the hands make a target angle 180^o, and the time it takes for the hands to make the target angle.

This problem is no other than the previous problem except the present time, which is 01:00:00.
Thus, the hour hand is 30^o ahead at the beginning, whereas the minute hand starts at the 12 o'clock marker, the same spot as in the previous problem.

So suppose now again, H is the angle made by the hour hand, M is the angle made by the minute hand, d is the angle between the two hands, and t is the time taken.

Then, if the current time is midnight, 00:00:00, or is noon, 12:00:00, we get $H = ht$, and $M = mt$, where h is the speed of the hour hand, and is 0.5, and m is the speed of the minute hand, and is 6. So we get $H = 0.5t$ and $M = 6t$.

In this problem, the present time is 01:00:00.
So the hour hand is 30^o ahead at the beginning, whereas the minute hand starts at the 12 o'clock marker, the same spot as in the previous problem.

Thus, M remains to be 6t, but H gets to be modified a little the way as follows. H = 30 + 0.5t.

Although the minute hand is behind the hour hand at the beginning, it will overtake the hour hand soon. So d = M – H is negative until the minute hand overlaps the hour hand, but will be positive, and get larger after the minute hand overlaps and overtakes the hour hand.
Thus, M is bigger than H when the target angle happens.
So the angle between the hands d is M – H, and has to be set to 180. Thus, we get

$d = 6t - (30 + 0.5t) = 180 \Rightarrow 6t - 30 - 0.5t = 180$

$\Rightarrow 5.5t = 180 + 30 \Rightarrow t = \dfrac{180+30}{5.5} = \dfrac{180}{5.5} + \dfrac{30}{5.5}$

So we can see that it takes $\frac{30}{5.5}$ minutes more for 180^o to happen than the case where the present time is midnight.

Let's now split the minutes into minutes and seconds

$$\frac{30}{5.5} = \frac{6}{1.1} = \frac{5.5+0.5}{1.1} = 5 + \frac{0.5}{1.1} = 5 + \frac{5}{11} \text{ minutes}$$

So we get $\frac{30}{5.5} = 5 + \frac{5}{11}$ minutes.

In seconds, we get $\frac{5}{11}$x 60 = $\frac{300}{11} = \frac{220+80}{11} = 20 + \frac{80}{11}$

$= 27 + \frac{3}{11}$ seconds.

Thus, it takes 5 minutes and $27 + \frac{3}{11}$ seconds more for 180^o to happen than the case where the present time is midnight.

Now, we have t = $\frac{180+30}{5.5} = \frac{180}{5.5} + \frac{30}{5.5}$, which is the time for 180^o to happen if the current time is 1 o'clock.

And we have $\frac{30}{5.5}$ minutes = 5 minutes and $27+\frac{3}{11}$ seconds,

and $\frac{180}{5.5}$ minutes = 32 minutes and $43+\frac{7}{11}$ seconds.

So we get t = $\frac{180}{5.5}+\frac{30}{5.5}$

= (5 + 32) minutes and (27 + 3/11 + 43 + 7/11) seconds

= 37 minutes and (70 + 10/11) seconds

= 37 minutes and (60 + 10 + 10/11) seconds

= 38 minutes and 10 + 10/11 seconds.

Therefore, 180^o happens at 01 : 38 : 10 + $\frac{10}{11}$.

Suggestions or Solutions
To The Problem 0.2 In Examples 7

0.2. If the current time is 14 o'clock (14:00:00), find the first moment of time when the hands make a target angle 180^o, and the time it takes for the hands to make the target angle.

Again, this is no other than the case where both hands are at the 12 o'clock marker. In this case, the hour hand is at the 2 o'clock marker. So the hour hand is 2 x 30^o ahead at the beginning.

So suppose now again, H is the angle made by the hour hand, M is the angle made by the minute hand, d is the angle between the two hands, and t is the time taken.

Then, if the present time is midnight, 00:00:00, we get
H = ht, M = mt, and d = M – H, where h is the speed of the hour hand, and is 0.5, and m is the speed of the minute hand, and is 6.

In this problem, the present time is 14:00:00.

So the hour hand is 2 x 30^{o} ahead at the beginning, whereas the minute hand starts at the same spot as in the previous problem.

Thus, M remains to be 6t, but H needs to be modified a bit as follows. H = 60 + 0.5t.

So again, d is negative until the overlap happens, but will be positive and increasing after the overlap.

And M is bigger than H when the target angle happens.

Thus, the angle between the hands d is still M – H, and has to be 180. So we get

$$d = 6t - (60 + 0.5t) = 180 \Rightarrow 6t - 60 - 0.5t = 180$$

$$\Rightarrow 5.5t = 180 + 60 \Rightarrow t = \frac{180+60}{5.5} = \frac{180}{5.5} + \frac{30}{5.5} + \frac{30}{5.5}$$

Thus, we can see that it takes $\frac{30}{5.5}$ minutes more for 180^{o} to happen than the case with the present time of 01:00:00 or 13:00:00.

And we have $\frac{30}{5.5}$minutes = 5 minutes and $27+\frac{3}{11}$seconds, and $\frac{180}{5.5}$minutes = 32 minutes and $43+\frac{7}{11}$seconds.

So we get t =$\frac{180}{5.5}+\frac{30}{5.5}+\frac{30}{5.5}$

= (10 + 32) minutes and (54 + 6/11 + 43 + 7/11) seconds

= 42 minutes and (97 + 13/11) seconds

= 42 minutes and (60 + 37 + 1 + 2/11) seconds

= 43 minutes and 38 + 2/11 seconds.

Therefore, 180^{o} happens at 14 : 43 : 38 +$\frac{2}{11}$.

Suggestions or Solutions
To The Problem 0.3 In Examples 7

0.3. If the current time is 7 o’clock (07:00:00), find the first moment of time when the hands make a target angle 180^o, and the time it takes for the hands to make the target angle.

Let’s have a look at the initial situation with the hands, and see how the hands are positioned at the beginning.

The hour hand is at the 7 o’clock marker, and the minute hand is at the 12 o’clock marker now. The target angle is 180^o. Then, we can see that the angle between the hands d is already bigger than the target angle now.

So we can expect the situation to be reversed the way as follows. Opposite of the situation in the previous cases, the minute hand will be behind the hour hand when the target angle happens. So H is bigger than M when the target angle happens. Therefore, the equation M – H = 180 needs to be changed to this: H – M = 180.

At the beginning, the hour hand is at the 7 o'clock marker, so it is 30^o x 7 = 30^o x 6 + 30^o = 180^o + 30^o
ahead of the 12 o'clock marker.
And the minute hand begins at the 12 o'clock marker.

In other words, the hour hand has $(180 + 30)^o$ gap and the minute hand has 0^o gap from the 12 o'clock marker. Also, we can see that it will take much shorter time for the hands to make the target angle.
So we can get the time t the way as follows.

d = H – M = 180 = {0.5t + (180 + 30)} – 6t = 180,
where 180 + 30 is the gap between the 12 o'clock marker and the 7 o'clock marker.
$\Rightarrow$ 0.5t – 6t = 180 – (180 + 30) = -30 $\Rightarrow$ -5.5t = -30
$\Rightarrow$ t = $\frac{-30}{-5.5} = \frac{30}{5.5}$ Thus, we can expect that the target angle will happen after 30/5.5 minutes.
And we have 30/5.5 = 5 + 5/11 minutes.
In seconds, we get 5/11 x 60 = 27 + 3/11 seconds
Therefore, the target angle happens at 07 : 5 : 27 + 3/11.

Suggestions or Solutions
To The Problem 0.4 In Examples 7

0.1. If the current time is 20 o'clock (20:00:00), find the first moment of time when the hands make a target angle 180^o, and the time it takes for the hands to make the target angle.

Let's again, take a look at the initial situation with the hands in the clock, and see how the hands are positioned at the beginning.

The hour hand is at the 8 o'clock marker, and the minute hand is at the 12 o'clock marker. The target angle is 180^o.

Then again, the angle between the two hands is bigger than the target angle now.

So again, similar to the situation in the previous case, the minute hand is behind the hour hand when the target angle happens as in the previous case.

Thus, the equation remains to be this: $H - M = 180$.

At the beginning, the hour hand is at the 8 o'clock marker, so it is $30^o \times 8 = 30^o \times 6 + 60^o = 180^o + 60^o$

ahead of the 12 o'clock marker.

And of course. the minute hand is at the 12 o'clock marker. So the hour hand has $(180 + 60)^o$ gap and the minute hand has 0^o gap from the 12 o'clock marker.

So we can get the time t the way as follows.

d = H – M = 180 = {0.5t + (180 + 60)} – 6t = 180,

where 180 + 60 is the gap between the 12 o'clock marker and the 8 o'clock marker.

$\Rightarrow$ 0.5t – 6t = 180 – (180 + 60) = -60 $\Rightarrow$ -5.5t = -60

$\Rightarrow t = \frac{-60}{-5.5} = \frac{60}{5.5}$. Thus, we can expect that the target angle will happen after $\frac{60}{5.5}$ minutes.

And we have $\frac{60}{5.5} = 10 + \frac{10}{11}$ minutes.

In seconds, we get $\frac{10}{11}$x 60 = $\frac{600}{11} = \frac{550+50}{11} = 50 + \frac{50}{11}$

$= 50 + \frac{44+6}{11} = 54 + \frac{6}{11}$ seconds

Therefore, the target angle happens at 20 : 10 : 54 + $\frac{6}{11}$.

Suggestions or Solutions
To The Problem 1 In Examples 7

1. How many times do the two hands make 180^o in a day?

A day has 24 hours. And we can partition a day into 4 time areas as follows.

[00:00:00 ~ 06:00:00], [12:00:00 ~ 18:00:00],
[07:00:00 ~ 12:00:00], and [19:00:00 ~ 00:00:00]

We need to work with either of two equations.
One of the two holds for the case where the current time is in the two time areas as follows.

[00:00:00 ~ 06:00:00], and [12:00:00 ~ 18:00:00].

So one is from midnight (0 o'clock) until 6 o'clock in the morning, and the other is from the noon (12 o'clock) until 6 o'clock in the evening (18 o'clock).

Note that we need to exclude both 6 o'clock on the hour and 18 o'clock on the hour.
If the current time is either 6 o'clock or 18 o'clock, the equation does not hold, but we can see that the angle between the hands is already the target angle 180^o.

Let's now begin with the two time areas as follows.
[00:00:00 ~ 06:00:00] and [12:00:00 ~ 18:00:00].

In those two time areas, if the current time is every hour on the hour, the angle between the hands is less than the target angle 180^o at the beginning.
And M is bigger than H when the target angle 180 happens.
So we basically use the equation as follows.
$d = M - H = 180$

Next, looking at the other two time areas,
[07:00:00 ~ 12:00:00], and [19:00:00 ~ 00:00:00], we can see that if the current time is every hour on the hour, the angle between the hands is greater than the target angle 180^o at the beginning.

And H is bigger than M when the target angle 180 happens. So we basically use the equation as follows.

d = H – M = 180.

Let's now start with the first two time areas as follows.

[00:00:00 ~ 06:00:00] and [12:00:00 ~ 18:00:00].

Every hour marker is 30^o away from the neighboring hour marker. So the equation is as follows:

$M – H = 180 \Rightarrow 6t – (30 \times n + 0.5t) = 180$,

where n = 0, 1, 2, … 5, if n = 0, the present time is 00:00:00 or 12:00:00, and if n = 1, the present time is 01:00:00 or 13:00:00, and so forth.

$\Rightarrow 6t – 30n – 0.5t = 5.5t – 30n = 180 \Rightarrow 5.5t = 180 + 30n$

$\Rightarrow t = 30(6 + n)/5.5 = 60(n + 6)/11$

$\Rightarrow t = 60(n + 6)/11$, where n = 0, 1, 2, … 5

Thus, the target angle happens 6 x 2 + 2 = 12 + 2 = 14 times in the two time areas as follows.

[00:00:00 ~ 06:00:00] and [12:00:00 ~ 18:00:00].

We are now left with the other two time areas as follows. [07:00:00 ~ 12:00:00] and [19:00:00 ~ 00:00:00].

The angle between the two hands is already greater than the target angle 180^o at the beginning, and H is bigger than M when the target angle happens, so we basically use the equation as follows. d = H – M = 180.

Every hour marker is 30^o away from the neighboring hour marker. And at 7 o'clock, the hour hand is $180^o + 30^o$ ahead of the minute hand. So the equation is as follows:

$H - M = 180 \Rightarrow (0.5t + 180 + 30 \times n) - 6t = 180$,
where n = 1, 2 . . . 5, if n = 1, the present time is 07:00:00 or 19:00:00, and if n = 2, the present time is 08:00:00 or 20:00:00, and so forth.

$\Rightarrow 0.5t + 30n - 6t = 0 \Rightarrow 5.5t = 30n \Rightarrow t = 30n/5.5 = 60n/11$
$\Rightarrow t = 60n/11$, where n = 1, 2,…, 5.
Thus, the target angle happens 5 x 2 times in the other two time areas.

So the angle of 180^o happens once for every hour interval. Thus, the hands make 180^o 24 times a day.

Note that both hands make 180^o at 6 o'clock on the hour or 18 o'clock on the hour. So if the present time is 06:00:00 or 18:00:00, 180^o happens for $t = 0$.

Suggestions or Solutions
To The Problem 2 In Examples 7

2. Find a formula if any. For instance, we may want to find a formula for each hour.

The final two equations we made in the problem 1 above can be the formulas.

We can use the equation (A) as a formula if the present time is in any of the two time areas as follows.
[00:00:00 ~ 06:00:00] and [12:00:00 ~ 18:00:00]

(A) $t = 60(n + 6)/11$,
where $n = 0, 1, 2 \ldots 5$, if $n = 0$, the present time is 00:00:00 or 12:00:00, and if $n = 1$, the present time is 01:00:00 or 13:00:00, and so forth.

And we can use the equation (B) as a formula if the present time is in any of the two time areas as follows.
[07:00:00 ~ 12:00:00] and [18:00:00 ~ 24:00:00].

(B) $t = 60n/11$,

where $n = 1, 2 \ldots 5$, if $n = 1$, the present time is 07:00:00 or 19:00:00, and if $n = 2$, the present time is 08:00:00 or 20:00:00, and so forth.

Note that the angle between the hands is 180^o at 06:00:00 or 18:00:00.