Examples A in Ratios and Rates

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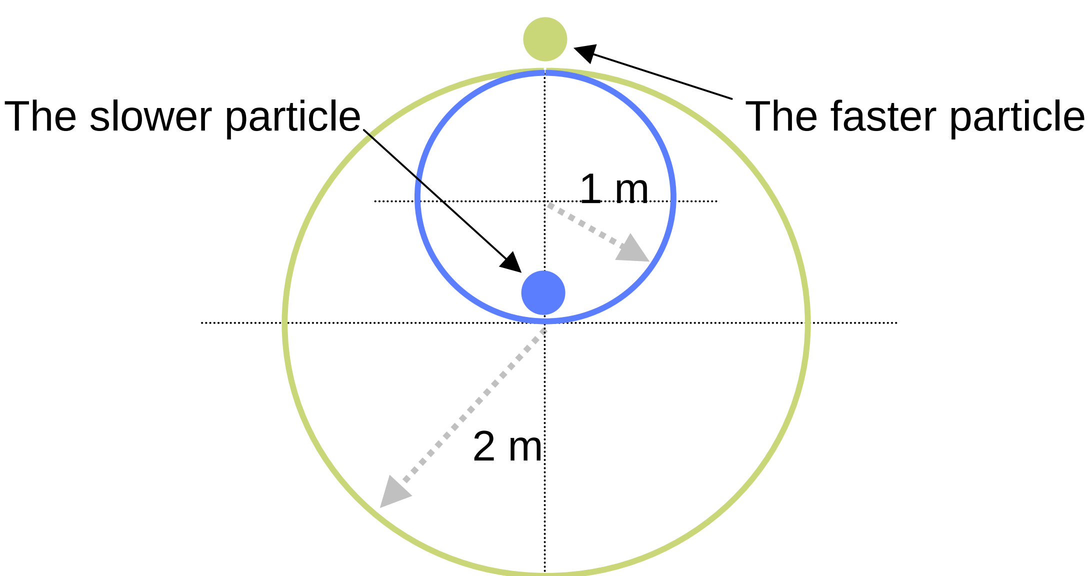
As in Fig. 0 below. one particle is orbiting at 10 m/s along a circular path of the radius 2 m, and the other particle is at 8 m/s along the other path with 1 m in radius. Each particle keeps moving in one direction. The smaller path is inscribing the larger, so both paths meet at one point. Suppose now that the faster particle starts at the position where the orbits meet, but that the slower one starts at the exact opposite of the point where the faster one starts. Then, see if they will meet. If not, what can we do to make the particles meet?

Fig. 0



We may want to begin with what this problem is made of. We can begin with examining the objects making up the problem. So to begin with, what are the objects?

They constitute the problem, and for instance, include the speeds of the particles and the radii of the orbits. Thus, make use of what's given in the problem description. Making use of it, we can make questions as follows.

Where can the particles meet? In other words, where will their meeting have to happen?
When can they meet?
When do they begin orbiting?
Where do they begin moving?
In what direction do they orbit?
Then, the possible answers can be as follows.

The meeting will have to happen at the point where the orbits meet.
They will meet at the same time, of course.
They begin orbiting at the same time.

The faster particle starts at the position where the orbits meet, but the other starts at the exact opposite of the point where the faster one starts, and of course, starts in its orbit. And each particle keeps moving in one direction.
Suppose now the point where the orbits meet is the target point. Then, if the particles meet, they get to the target point at the same time. How then do they get there concurrently?

Let’s now keep track of movements of the particles. Suppose first, when one particle arrives at the target point, the other is not there. What then does the one have to do?

Of course, it has to keep moving until it meets the other. How then do they move along their orbits?

The particles move at their speeds, but get to the target point at the same time if they meet, of course. How much time then do they spend until the meeting?

The same amount of time. At different speeds though. What then is different?

The distances covered. How then do we get the distances?

We can express the distance each particle has to cover along its orbit until it meets the other.
What then does the distance covered have to do with?

The speed and the time taken. For each particle thus, we have three objects to work on. One is the distance covered, another is the speed, and the other is the time taken to cover the distance until the meeting. We have another way though, to get the distance to be covered until the meeting in this case. How?

The particle can cover the distance the way as follows, too.

It moves along the track, and gets to the target point. If the other particle is not there at the time of its arrival at the target point, it continues moving, and gets to the target point again. And it moves along the same path over and over until the meeting. What then does the particle make?

Laps. And of course, the same is true to the other one, too. So the particles keep making their laps while orbiting. Thus, we want to find how many laps they have to make until the rendezvous, and then, we can express the distance.
How then can we get the numbers of laps?

We may want to first set up a couple of objects representing the number of the laps to be made. So suppose now,
the faster particle is A, the slower particle is B,
the number of the laps A makes = u,
and the number of the laps B makes = v.

Then, A begins orbiting at the target point,
and should make complete laps if it meets B.
However, B should not make complete laps if it meets A.

That's because every time B makes a complete lap, it does not end up with the target point. So it looks like we have a little problem with the laps the particles will have to make.
What then do we have a problem with?
Is it the number of laps?

Not quite! It's not the number of laps yet. We have a problem with the lap itself prior to the number of laps.
What lap then?

At the beginning, B has to make half a complete lap to get to the target point. So if B keeps making complete laps all the way, it never meets A. Thus, there is no point counting the number of complete laps B makes.

Nonetheless, we still need to use the number of laps to describe the distance to be covered until they meet.
What laps then?

Half laps.
That's because B has to get to the point of rendezvous, and the distance B covers until the meeting can be put in terms of a number of half laps. We can think thus, both are making half laps until they meet at the target point.
Then, we now want to know how many half laps both will have to make until they meet. Then, we want to consider two facts as follows.

Fact 1.
A will have to make an even number of the laps whereas B will have to make an odd number of the laps until both meet.
Fact 2.
They begin orbiting at the same time. They will meet at the same time. Thus, both will spend the same amount of time until the meeting. And each moves at its speed.

So suppose now that the same amount time = t.
Then, what can we draw from the two facts above?

They need to make their particular numbers of the laps in the time t until the meeting. So overall, we have the setting as follows.
Until the meeting, A will make a particular even number of its laps in t, and B will make a particular odd number of its laps in t.

So we can use the numbers of the laps, and use the amount of time t to be taken until the meeting, along with the speeds. How?

A will cover a particular distance in a particular number of the laps in time t. Also, A will cover the same distance orbiting at its speed in time t. And the same applies to B, too.

So for each particle, we can describe the distance covered at each speed in t, and the same distance is covered by the number of the laps in t.

We know the distance = the speed x the amount of time.
So using the speeds and the time, we can get the distances.
Suppose now F is the distance covered in t by the particle A, and S is the distance covered in t by the particle B.
Then, we get F = 10t, and S = 8t.

Let's now move on to the use of the laps. What then does the number of the laps have to do with the distance?

We know that each particle has to make a particular number of laps (u or v) until the meeting, and that each lap takes up half the length of an orbit, since it's half a complete lap. And we have the information on the lengths of both orbits.

So we can describe each distance using the number of the laps and the length of each lap. Then, each distance is the number of the laps x the length of the lap.
We have u = the number of the laps A makes, and v = the number of the laps B makes until the meeting.

So F = u x half the length of the orbit of A,
and S = v x half the length of the orbit of B

And assuming C is the circumference of a circle, D is the diameter, and r is the radius, we get $C = \pi D = \pi \times 2 \times r = 2\pi r$.

Each lap is half the amount of a complete lap.
So the distance covered by each lap = $2\pi r/2 = \pi r$.

We know F is the distance covered in t by the particle A, the radius of the orbit is 2 meters, and the number of the laps made until the meeting is u.
So $F = u \times \pi r = u \times \pi \times 2 = 2\pi u$, which is the distance A covers after its u laps in time t.

We know S is the distance covered in t by the particle B, the radius of the orbit is 1 meter, and the number of the laps made until the meeting is v.

So $S = v \times \pi r = v \times \pi \times 1 = \pi v$, which is the distance B covers after its v laps in t.

Thus, we have $F = 2\pi u$ and $S = \pi v$.

And we have $F = 10t$, and $S = 8t$, too.

We know each particle makes u or v laps in t.

So the distance covered at 10 m/sec in t by A is the same as the distance covered by u laps made by A.

And the distance covered at 8 m/sec in t by B is the same as the distance covered by v laps made by B.

Thus, we have $10t = 2\pi u$ and $8t = \pi v$.

So we now have 2 equations, but the equations have 3 unknowns, which means many solutions. The solution is not unique. What then can we do?

We can take the ratio between u and v.
So taking the ratio, we can get first,
$10t = 2\pi u \Rightarrow u = 10t/2\pi = 5t/\pi$, and $8t = \pi v \Rightarrow v = 8t/\pi$.
Thus next, we get $u:v = (5t/\pi):(8t/\pi) = 5:8$, since t/π is a common divisor. So we get $u:v = 5:8$.

We now need to find specific values for u and v in accordance with the ratio, and based on the situation where the particles orbit.

The particle A has to make an even number of its laps to get to the target point. So u has to be an even number. And u can be even if we double the entries in the ratio as in this: $u:v = 5:8 = 10:16$.

The particle B has to make an odd number of its laps to get to the target point. So v has to be odd. However, v cannot be odd, since any integer multiple of an even number cannot be odd.
Therefore, we can conclude that the particles will not meet.
What then can we do to make the particles meet?

What matters making the rendezvous?

The speeds of the particles, the lengths of the orbits, and the initial positions matter.

That’s because the ratio between the numbers of laps depends on speeds or lengths of orbits, along with initial positions.

We can get the ratio using the descriptions of the distances as follows.
the speed x the time taken = the length of a lap x the number of laps.

In the descriptions above, we can change the speed or the length of the lap. If the speed or the length changes, the time will change, of course. What then can we do to make the particles meet?

We need to change the ratio. If the ratio of u to v is an even integer over an odd integer, they can meet.

Therefore, we want to adjust the speeds or the lengths of the orbits of the particles accordingly.
Suppose for instance, we reset the speeds the way as follows keeping the track length unchanged.
The speed of the particle A = 16 m/s.
The speed of the particle B = 9 m/s.

We know that the speed x the time taken = the length of the lap x the number of the laps.

And we know the length of a lap of A is 2π, and the length of a lap of B is π.
So we can now quickly get two equations as follows.
$16t = 2\pi u$, and $9t = \pi v$.

Then, we get $u = 16t/2\pi = 8t/\pi$, and $v = 9t/\pi$.
So we get $u:v = 8:9$.

Then, after the particle A makes 8 laps and the particle B makes 9 laps, they meet at the target point where the tracks meet, and the lap is half a complete lap.

Let’s now check with the time, the value of t, it will take for the first meeting to happen.
Then, since it’s the first meeting, we set $u = 8$.
So we get $16t = 2\pi u = 16\pi \Rightarrow t = \pi$ seconds.

Next, let’s make sure if it takes the same amount of time for the other to make 9 laps.
Then, since it’s the first meeting, we set $v = 9$.
Then, we get $9t = \pi v = 9\pi \Rightarrow t = \pi$ seconds.

Thus, after π (approximately 3.14) seconds, the particles will meet for the first time. Then, is that it?

Not quite! What about the next meeting, and all the subsequent meetings?

For the subsequent meetings, do we just have to work on nothing but the ratio between u and v? In other words, will the particles meet every 8 and 9 laps for each of all their meetings? So will the particles meet in the same time interval of π seconds?

No, that is not the case. That's because if the particle B makes another set of 9 laps, it cannot get to the target point, since it will be at the point opposite of the target point. We can use the same ratio though, but the particle B should make complete laps for the subsequent rendezvouses.

Once they've met for the first time, they are in the situation where they start orbiting at the target point where the tracks meet. So the lap is now the complete lap, and the particles will meet every time each particle makes its particular number of the complete laps.
Thus, the situation is the same as the one in the example 9. Do we then need to set up all the equations again?

We don't need to. The ratio remains the same, and the only thing that gets changed is the time. It's because not just one, but all particles need to now make complete laps.

From the second time on, one particle will meet the other in the same number of complete laps as the number of the half laps taken for it to meet the other.

Thus, we just have to increase the respective length in each of the laps by the same scale.

Until the first meeting, we have $16t = 2\pi u$, and $9t = \pi v$.

The distance each particle covers until the meeting is
the speed x the time = the lap length x the number of laps.

The speed of the particle A is 16 m/s, the number of the laps by A = u, and the radius is 2.
The speed of the particle B is 9 m/s, the number of the laps by B = v, and the radius is 1. Then, we get

$16t = 2\pi r \times u \Rightarrow 16t = 4\pi u \Rightarrow 8t = 2\pi u \Rightarrow u = 8t/2\pi$.
$9t = 2\pi r \times v \Rightarrow 9t = 2\pi v \Rightarrow v = 9t/2\pi$. So we get $u:v = 8:9$.

Thus, we just have to increase the respective length in each of the laps by the same scale.
Thus, every time the particle A makes 8 complete laps, and the particle B makes 9 complete laps, every subsequent rendezvous happens.

And let's now see the amount of time it takes for the subsequent meeting to happen.
Then, setting $u = 8$ for A, we get
$u = 8t/2\pi = 8 \Rightarrow t/2\pi = 1 \Rightarrow t = 2\pi$
Next, setting $v = 9$ for B, we get
$v = 9t/2\pi = 9 \Rightarrow t/2\pi = 1 \Rightarrow t = 2\pi$, too.

Thus, after the first rendezvous, the particles will meet every 2π seconds subsequently.
The particles will be able to meet depending on the speeds of themselves or the lengths of their tracks.
We should not take entries in a ratio for just numbers, and should note that a ratio specifies a relationship in quantity.
Can we say then that the meeting is periodic?

Of course, we can, and we can say that after the first meeting, the period is 2π seconds in time, 8 laps for A, and 9 laps for B.

It only takes π seconds for the first meeting to happen, but it takes 2π seconds for every subsequent meeting to happen.

It takes 8 half laps for A, and 9 half laps for B for the first meeting to happen, but it takes 8 complete laps for A, and 9 complete laps for B for every subsequent meeting to happen. How about resetting the lengths of the tracks?

Let's now, make an adjustment on the lengths of the tracks. We can reset them by changing the radii.
The initial situation was as follows.

The number of the laps A makes = u, the number of the laps B makes = v, A is at the target point where the two orbits meet, and B is at the point the opposite of the target point, the speed of A = 10 m/s, the speed of B = 8 m/s, the time the particles have to spend until they meet = t, and one lap is half a complete lap.

And now, we reset the lengths of the orbits the way as follows. The radius of the orbit A is on is 1 meter.
The radius of the orbit B is on is 8/15 meter.
Then, we get $10t = \pi r \times u = \pi u$ and $8t = \pi r \times v = (8/15)\pi v$.
So we get $u = 10t/\pi$ and $v = 15t/\pi$.

Thus, we get $u:v = 10:15 = 2:3$.

So if the particle A makes 2 laps, and the particle B makes 3 laps, the first rendezvous happens.

And let's now see the amount of time it takes for the first meeting to happen. Then, setting $u = 2$ for A, we get

$u = 10t/\pi = 2 \Rightarrow 5t/\pi = 1 \Rightarrow t = \pi/5$.

Next, setting $v = 3$ for B, we get

$v = 15t/\pi = 3 \Rightarrow 5t/\pi = 1 \Rightarrow t = \pi/5$, as well.

So the first meeting happens after $\pi/5$ second.

Is there any other way to solve the problem?

We can produce equations using complete laps instead of half laps.

Then, we set the number of laps the way as follows.

The number of the laps A makes = u,

the number of the laps B makes = $v + 1/2$.

So B makes v complete laps and a half lap, and A makes u complete laps.

And the situations is as follows.

A starts at the target point where the two orbits meet,
B starts at the point the opposite of the target point, and is on its orbit, of course, the speed of A is 16 m/s,
the speed of B is 9 m/s,
the time the particles have to spend until they meet is t,
the radius of the orbit A is on is 2 meters,
and the radius of the orbit B is on is 1 meter.

We know a complete lap covers the distance of $2\pi r$, which is the length of the orbit. So we get for A, $16t = 2\pi ru$, and for B, we get $9t = 2\pi r(v + 1/2)$.
Thus, after A makes u laps, and B makes $(v + 1/2)$ laps, both meet at the target point.

Then, for A, we get $16t = 2\pi ru = 4\pi u$, since $r = 2$, so we get $u = 8t/2\pi$, where u is a positive integer.
And for B, we get $9t = 2\pi r(v + 1/2) = 2\pi(v + 1/2)$, since $r = 1$, so we get $v + 1/2 = 9t/2\pi$, where v is a non-negative integer.

Thus, we get $u:(v + 1/2) = 8:9 \Rightarrow u/(v + 1/2) = 8/9$
$\Rightarrow 9u = 8(v + 1/2) = 8v + 4 \Rightarrow 9u = 8v + 4$

So we can get u = 4 and v = 4.
Thus, after A makes 4 complete laps and B makes 4 and a half complete laps, they meet for the first time. In other words, after A makes 8 half laps and B makes 9 half laps, they meet for the first time.
What then about the next meeting, and all the subsequent meetings?

Once they've met for the first time, they are in the situation where both start orbiting at the point where the tracks meet. Thus, the situation is the same as the one in the example 9.

The ratio remains the same, and the only thing that gets changed is the time. It's because no particle makes any partial lap, so all particles now make complete laps only.

So assuming after A makes u laps, and B makes v laps, both meet at the target point, we get u:v = 8:9.
Let's see now if it really is the case.
The speed of A is 16 m/s,
the speed of B is 9 m/s,

the time the particles have to spend until they meet is t,
the radius of the orbit A is on is 2 meters,
and the radius of the orbit B is on is 1 meter.
Then, we get
$16t = 2\pi ru = 4\pi u$ and $9t = 2\pi rv = 2\pi v \Rightarrow 18t = 4\pi v$
$u:v = 16t:18t = 8:9$.

Thus, every time the particle A makes 8 complete laps, and the particle B makes 9 complete laps, every subsequent rendezvous happens.
What then about this ratio: $u:(v + 1/2) = 8:9$?

We can use it, too, so using it, we can still find the laps and the time for the subsequent meetings.

Without using $u:v = 8:9$ though, it's not very easy to use this ratio: $u:(v + 1/2) = 8:9$.

We get $u:(v + 1/2) = 8:9 \Rightarrow u/(v + 1/2) = 8/9$
$\Rightarrow 9u = 8(v + 1/2) = 8v + 4 \Rightarrow 9u = 8v + 4$
So after some trial and error, we can get $u = 4$ and $v = 4$.

It takes however, a lot of trial and error to find the values of u and v for the next meeting.

Using $9u = 8v + 4$, we can get these:

$u = 12$ and $v = 13$ for the second meeting,

and $u = 20$ and $v = 22$ for the third.

By trial and error, it's not easy to see this:

$9u = 8v + 4 \Rightarrow 9 \times 12 = 8(13 + 1/2)$, and $9 \times 20 = 8(22 + 1/2)$

Using the ratio $u:v = 8:9$, we can quickly see that every subsequent meeting happens every time A makes 8 laps and B makes 9 laps after the first meeting.