Examples B in Ratios and Rates

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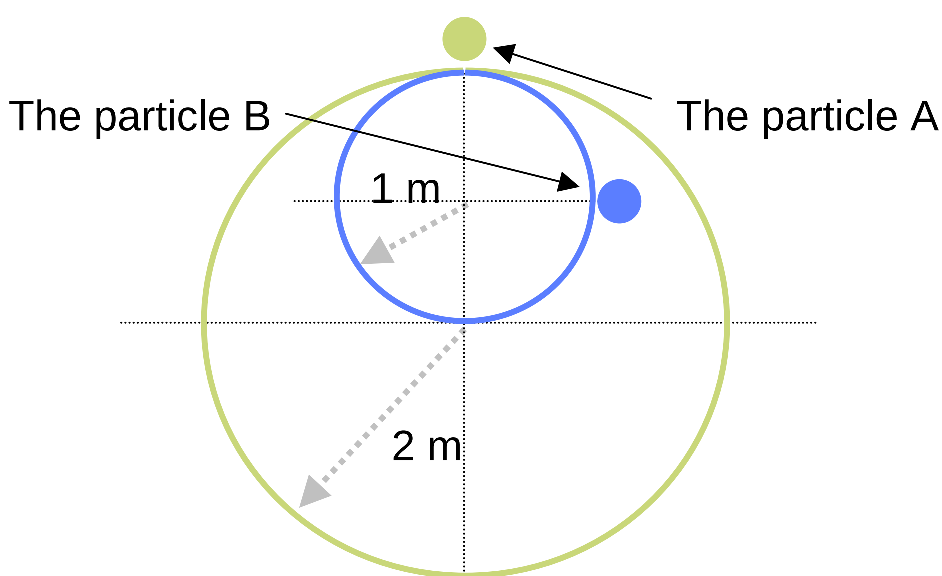
As in Fig. 0 below, one particle is orbiting at 10 m/s along a circular path of the radius 2 m, and the other particle is at 8 m/s along the other path with 1 m in radius. Both are moving clockwise. The smaller path is inscribing the larger, so both paths meet at one point. Suppose A starts at the point where the tracks meet, and B starts at the point a quarter track past to the right of the point where the tracks meet. Then, see if the two particles will meet. If not, what can we do to make the particles meet?

Fig. 0



Let’s begin with what this problem is made of.
We can begin with examining the objects making up the problem. So to begin with, what are the objects?

They constitute the problem, and for instance, include the speeds of the particles and the radii of the orbits. Thus, make use of what’s given in the problem description.
Making use of it, we can make questions as follows.

Where can the particles meet? In other words, where will their meeting have to happen?
When can they meet?
When do they begin orbiting?
Where do they begin moving?
In what direction do they orbit?
Then, the possible answers can be as follows.

The meeting will have to happen at the point where the orbits meet.
They will meet at the same time, of course.
They begin orbiting at the same time.

The faster particle starts at the position where the orbits meet, but the other starts at the point a quarter orbit past to the right of the point where the orbits meet., and of course, starts in its orbit. And both orbit clockwise.

Suppose now that the point where the orbits meet is the target point. Then, if the particles meet, they both get to the target point at the same time.
How then do they get there concurrently?

Let’s now keep track of movements of the particles. When one particle arrives at the target point, if the other is not there, it has to keep moving until it meets the other.

The particles move at their speeds, but meet at the same time if they meet, of course. They spend the same amount of time until the meeting. At different speeds though.
What then is different?

The distances covered. How then do we get the distances?

We can express the distance each particle covers along the orbit until it meets the other.
What then does the distance covered have to do with?

The speed and the time taken. For each particle thus, we have three objects to work on. One is the distance covered, another is the speed, and the other is the time taken to cover the distance until the meeting. We have another way though, to get the distance to be covered until the meeting in this case.　How?

Each particle covers the distance the way as follows, too.
It moves along the track, gets to the target point, and if the other is not there, it continues moving, and gets to the target point again. And it moves along the same path over and over until the meeting.　What then does the particle make?

Laps. And of course, the same is true to the other, too. So the particles keep making laps while orbiting.
Thus, we want to find how many laps each particle makes until the rendezvous, and then, we can express the distance.

How then can we get the numbers of laps?

First, we need to set up objects that represent the number of the laps to be made. So suppose now,
the number of the laps A makes = u,
and the number of the laps B makes = v.

Then, A begins orbiting at the target point,
and should make complete laps if it meets B.
However, B should not make complete laps if it meets A.

That's because every time B makes a complete lap, it does not end up with the target point. So it looks like we have a little problem with the laps the particles will have to make. What lap then?

At the beginning, B will have to make three quarters of a complete lap to get to the target point.
So if B keeps making complete laps all the way, it will never meet A. What laps then?

Quarter laps.

That's because B has to get to the point of rendezvous, and the distance B covers until the meeting can be put in terms of a number of quarter laps. So we can think that both make quarter laps until they meet at the target point.

Thus, we now need to know how many quarter laps both will make until they meet. Then, we need to consider two facts as follows.

Fact 1.

A will have to make an even number of the laps whereas B will have to make an odd number of the laps until both meet.

Fact 2.

Both will spend the same amount of time until the meeting, and each moves at its speed.

So suppose now that the same amount time = t.

Then, what can we draw from the two facts above?

Each particle needs to make a particular numbers of its laps in the amount of time t until the meeting. So overall, we have the setting as follows.

Until the meeting, A makes a particular even number of laps in t, B makes a particular odd number of laps in t, and each moves at its speeds.

So we can express each distance to be covered by each particle until the meeting using the number of its laps, and can express that distance using its speed, too.

A covers a particular distance in a particular number of its laps in time t. Also, A covers the same distance orbiting at its speed in time t. And the same applies to B, too.

We have these:
The number of the laps A makes = u.
The number of the laps B makes = v.
The speed of A = 10 m/s. The speed of B = 8 m/s.
The amount of time both particles will have to spend until they meet = t.
The radius of the track A is on = 2 meters.
The radius of the track B is on = 1 meter.

Then, for each particle, we can express the distance covered at each speed in t, and also, can express the same distance covered by the number of its laps in t.

We know the distance = the speed x the amount of time.
So using the speeds and the time, we get the distances.

Thus, suppose now F is the distance covered in t by A, and S is the distance covered in t by B.
Then, we get F = 10t, and S = 8t.

Let’s now move on to the use of the laps. What then does the number of the laps have to do with the distance?

Each particle makes a particular number of laps (u or v) until the meeting, and each lap takes up a quarter of the length of the orbit, since it’s a quarter of a complete lap. And we have the information on the lengths of both orbits.
So we can express each distance using the number of the laps and the length of each lap, because each distance is the number of the laps x the length of the lap.

We have u = the number of the laps A makes, and v = the number of the laps B makes until the meeting.

So F = u x a quarter of the length of the orbit of A,
and S = v x a quarter of the length of the orbit of B

And assuming C is the circumference of a circle, D is the diameter, and r is the radius, we get $C = \pi D = \pi \times 2 \times r = 2\pi r$.

Each lap is a quarter of a complete lap.
So the distance covered by each lap = $2\pi r/4 = \pi r/2$.

We know F is the distance covered in t by the particle A, the radius of the track is 2 meters, and the number of the laps made until the meeting is u.
So $F = u \times \pi r/2 = u \times \pi \times 2/2 = \pi u$, which is the distance A covers after its u laps in time t.

We know S is the distance covered in t by the particle B, the radius of the track is 1 meter, and the number of the laps made until the meeting is v.

So $S = v \times \pi r/2 = v \times \pi \times 1/2 = \pi v/2$, which is the distance B covers after its v laps in t.

Thus, we get $F = \pi u$ and $S = \pi v/2$.

And we have $F = 10t$, and $S = 8t$.

We know each particle makes u or v laps in t.

So the distance covered at 10 m/sec in t by A is the same as the distance covered by u laps made by A.

And the distance covered at 8 m/sec in t by B is the same as the distance covered by v laps made by B.

Thus, we have $10t = \pi u$ and $8t = \pi v/2$.

So we now have 2 equations, but the equations have 3 unknowns, which means many solutions. The solution is not unique. What then can we do?

We can take the ratio between u and v. So taking the ratio, we can get first, $10t = \pi u \Rightarrow u = 10t/\pi$, and $8t = \pi v/2$ $\Rightarrow v = 16t/\pi$. Thus next, we get $u:v = (10t/\pi):(16t/\pi) = 5:8$, since $2t/\pi$ is a common divisor. So we get $u:v = 5:8$.

We now need to find specific values for u and v in accordance with the ratio, and based on the situation where the particles orbit.
The particle A has to make an even number of its laps to get to the target point. Then, u has to be even. And u can be an even number if we double the entries in the ratio.
The particle B has to make an odd number of its laps to get to the target point, so v has to be odd, but cannot be odd, since any integer multiple of an even number cannot be odd.
Thus, the meeting cannot happen.
What then can we do to make the particles meet?

What matters making the rendezvous?

The speeds of the particles, the lengths of the tracks, and the initial positions matter.
It’s because the ratio between the numbers of laps depends on speeds or track lengths, along with initial positions. We can get the ratio using the descriptions of the distances as follows. the speed x the time taken = the length of a lap x the number of laps.

In the descriptions above, we can change the speed or the length of the lap. If the speed or the length changes, the time will change, of course. What then can we do to make the particles meet?

We need to change the ratio. If the ratio of u to v is an even integer over an odd integer, and also, if u is a multiple of 4 and v is 3 and a multiple of 4, they can meet.
Therefore, we need to adjust the speeds or the lengths of the orbits accordingly.

Suppose for instance, we reset the speeds the way as follows keeping the track length unchanged.
The speed of the particle A = 16 m/s.
The speed of the particle B = 4.5 m/s.

We know that the speed x the time taken = the length of the lap x the number of the laps. And we know the length of a lap of A is π, and the length of a lap of B is $\pi/2$.
So we can now quickly get two equations as follows.
$16t = \pi u$, and $4.5t = \pi v/2$.

Then, we get u = $16t/\pi$, and v = $9t/\pi$. So we get u:v = 16:9. Thus, after the particle A makes 16 laps and the particle B makes 9 laps, do they meet at the target point where the tracks meet, and the lap is a quarter of a complete lap?

No, it doesn't look like it. Why not?

A can get to the target point after 16 laps.
However, B does not get to the target point after 9 laps, because B will end up arriving at the point opposite of the target point, so B cannot meet A.

So the reset does not seem to make any progress at all.
If we go ahead and make a conclusion that the reset does not work however, then we are a bit too much in a hurry.

A ratio is a statement that explains the relationship in quantity between objects. A ratio specifies the relationship only. So a ratio does not just specify particular values, but it specifies groups of values, too. Thus, we can have this:
u:v = 16:9 = 32:18 = 48: 27 = 64:36 = 8:4.5 = 1.6:0.9 = …

So we can triple the entries in the ratio.
Then, we get $u = 3 \times 16 = 48$, and $v = 3 \times 9 = 27$, so we can now see that A gets to the target point after 48 laps, and that B gets to the target point after 27 laps.

Let's now check with the time, the value of t, it will take for the first rendezvous to happen.
Then, since it's the first rendezvous, we set $u = 48$.
Then, we get $16t = \pi u = 48\pi \Rightarrow t = 3\pi$ seconds.
Next, let's make sure if it takes also, the same amount of time for the other to make 27 laps.
Then, since it's the first rendezvous, we set $v = 27$.
Then, we get $4.5t = \pi v/2 = 27\pi/2 \Rightarrow t = 3\pi$ seconds, too.
Thus, after 3π (approximately 9.42) seconds, the particles will meet for the first time.
Let's now proceed with the next meeting, and all the subsequent meetings. Do we then just keep working with the ratio found above?

The ratio still works, but the lap doesn't. It's because the particles are in a different situation. Where are they now?

They are at the target point where the orbits meet, and are about to begin orbiting at the same time for the next meeting.

So the situation is the same as the one in the Example 9.
Thus, the lap is back to the complete lap, and then, the particles will meet every time each particle makes its particular number of its complete laps.

From the second time on, each lap is a complete lap, but the ratio is the same. It's because we just have to increase the length of a lap of each particle by the same scale.

The lap is now a complete lap, and the scale is 4.
That's because the lap length is now 4 times the length of a quarter of a complete lap.
And we had $16t = \pi u$, and $4.5t = \pi v/2$.
So we now have $16t = 4\pi u$, and $4.5t = 4\pi v/2$.
Thus, for the particle A, we get $16t = 4\pi u \Rightarrow u = 16t/4\pi$.
And for the particle B, we get $4.5t = 4\pi v/2 \Rightarrow v = 9t/4\pi$
So we get $u:v = 16:9$.

Every time thus, A makes 16 complete laps, and B makes 9 complete laps, every subsequent rendezvous happens.
Let's find now the amount of time it takes for the subsequent meeting to happen. Setting then first, $u = 16$ for A, we get
$16t = 4\pi u = 4\pi \times 16 \Rightarrow t = 4\pi$.
And setting next, $v = 9$ for B, we get
$4.5t = 4\pi v/2 = 4\pi \times 9/2 = 4\pi \times 4.5 \Rightarrow t = 4\pi$, too.
So after the first rendezvous, the particles will meet every 4π seconds subsequently.

The particles will be able to meet depending on the speeds of themselves or the lengths of their tracks, along with their initial positions.
We should not take entries in a ratio for just numbers, and should note that a ratio specifies a relationship in quantity.
Can we say then that the meeting is periodic?

Of course, we can, and we can say that after the first meeting, the period is 4π seconds in time, 16 laps for A, and 9 laps for B. And each lap is a complete lap.

It takes 3π seconds for the first meeting to happen, but it takes 4π seconds for every subsequent meeting to happen.

It takes 48 quarter laps for A, and 27 quarter laps for B for the first meeting to happen, but it takes 16 complete laps for A, and 9 complete laps for B for every subsequent meeting to happen.

Let's now, make adjustments on the lengths of the tracks.
We can reset them by changing the radii in this case.
The initial situation was as follows.
The speed of A = 10 m/s, the speed of B = 8 m/s, and one lap is a quarter of a complete lap.

And now, let's reset the lengths of the tracks the way as follows, and see if the particles meet.
The radius of the track A is on = 2 meters
The radius of the track B is on = 16/15 meters
Then, we get $10t = 2\pi r/4 \times u = \pi u$, since $r = 2 \Rightarrow 10t = \pi u$.
And we get $8t = 2\pi r/4 \times v = 2\pi(16/15)/4 \times v$, since $r = 16/15$.
So we get $8t = \pi(32/15)/4 \times v = (8/15)\pi v \Rightarrow 8t = (8/15)\pi v$.

Thus, we get $u = 10t/\pi$ and $v = 15t/\pi$.
So we get $u:v = 10:15 = 2:3$.
Thus, if the particle A makes 2 laps, and the particle B makes 3 laps, does the first rendezvous happen?

No, it doesn't. If it does, u has to be a multiple of 4, and v has to be 3 and a multiple of 4. With the ratio $u:v = 2:3$, it not possible for them to be so. Thus, the particles never meet.
What then can we do to make the particles meet?

We make the ratio work. If the ratio of u to v is an even integer over an odd integer, and also, if u is a multiple of 4 and v is 3 and a multiple of 4, they can meet. So we need to adjust the speeds or the lengths of the orbits accordingly.

Suppose this time, we reset the track lengths the way as follows keeping the speeds unchanged.
The radius of the orbit A is on = R meters.
The radius of the orbit B is on = r meters.
Then, we get $10t = 2\pi R/4 \times u = 2\pi Ru/4 \Rightarrow 20t = \pi Ru$.
And we get $8t = 2\pi r/4 \times v = 2\pi rv/4 \Rightarrow 16t = \pi rv$.

So we get

$20t = \pi Ru \Rightarrow u = 20t/\pi R$, and $16t = \pi rv \Rightarrow v = 16t/\pi r$.

Thus, we get

$u:v = 5/R : 4/r \Rightarrow u/v = (5/R)/(4/r) = 5r/4R \Rightarrow u:v = 5r:4R$

Now, if $u:v = 4:3$, the meeting happens, because after A makes 4 laps and B makes 3 laps, they are at the target point at the same time.

We have $u:v = 5r:4R$.

So if we set $5r:4R = 4:3$, the meeting happens.

Thus, now resetting the radii, we can set for instance,

$5r:4R = 4:3 = 40:30$. Then, we get

$5r = 40 \Rightarrow r = 40/5 = 8$ and $4R = 30 \Rightarrow R = 30/4 = 7.5$

So we can reset the radii the way as follows.

The radius of the track A is on = 7.5 meters

The radius of the track B is on = 8 meters

Then, we get $10t = 2\pi r/4 \times u = 15\pi u/4$, since $r = 7.5$

$\Rightarrow 40t = 15\pi u \Rightarrow 8t = 3\pi u \Rightarrow u = 8t/3\pi$

And we get $8t = 2\pi r/4 \times v = 2\pi \times 8/4 \times v$, since $r = 8$.

So we get $8t = 4\pi v \Rightarrow v = 8t/4\pi$

So we get u:v = $(8t/3\pi):(8t/4\pi) = (1/3):(1/4) = 4:3$.

Thus, if the particle A makes 4 laps, and the particle B makes 3 laps, the first rendezvous happens.

And let's now see the amount of time it takes for the first meeting to happen. Then, setting $u = 4$ for A, we get

$u = 8t/3\pi = 4 \Rightarrow 2t/3\pi = 1 \Rightarrow t = 3\pi/2$ sec

Next, setting $v = 3$ for B, we get

$v = 8t/4\pi = 3 \Rightarrow 2t/\pi = 3 \Rightarrow t = 3\pi/2$ sec, as well.

So the first meeting happens after $3\pi/2$ seconds.

What then about the subsequent meetings?

The situation is the same as that of example 9.

So the ratio above remains the same, so u:v = 4:3, but the lap is a complete lap. Thus, every time A makes 4 complete laps and B makes 3 complete laps, their meeting happens.

Checking to see if it really is the case, we get

$10t = 2\pi r \times u = 15\pi u$, since $r = 7.5$

$\Rightarrow 10t = 15\pi u \Rightarrow 2t = 3\pi u \Rightarrow u = 2t/3\pi$

And we get $8t = 2\pi r \times v = 2\pi \times 8 \times v$, since $r = 8$.

So we get $t = 2\pi v \Rightarrow v = t/2\pi$

Thus, we get $u:v = (2t/3\pi):(t/2\pi) = (2/3):(1/2) = 4:3$

Is there any other way to solve this problem?

We can produce equations in terms of complete laps instead of quarter laps. A complete lap covers the length of each orbit, and is $2\pi r$ in this case.

And the situation is as follows.

The number of the laps A makes = u,

the number of the laps B makes = v,

A is at the point where the two tracks meet, and B is at the point a quarter orbit past to the right of the target point where the orbits meet,

the speed of A = 16 m/s, the speed of B = 4.5 m/s,

the time the particles have to spend until they meet = t,

the radius of the track A is on = 2 meters, and

the radius of the track B is on = 1 meter.

Then, for A, we get $16t = 2\pi ru = 4\pi u$, since $r = 2$, so we get $u = 8t/2\pi$.

And for B, we get

$4.5t = 2\pi r(v + 3/4) = 2\pi(v + 3/4)$, since $r = 1$.

$\Rightarrow 4.5t = 2\pi(v + 3/4) \Rightarrow 9t = 4\pi(v + 3/4)$.

So we get $v + 3/4 = 9t/4\pi$.

And we have $u = 8t/2\pi = 16t/4\pi$.

Thus, we get $u:(v + 3/4) = 16:9 = 48:27 = 12:(6 + 3/4)$

$\Rightarrow u:(v + 3/4) = 12:(6 + 3/4)$

So we can get $u = 12$ and $v = 6$.

And we can get the same this way, too:

$u:(v + 3/4) = 16:9 \Rightarrow u/(v + 3/4) = 16/9 \Rightarrow 9u = 16v + 12$

So we can get $u = 12$ and $v = 6$.

Both of the two methods above requires trial and error. And finding u and v for the subsequent meetings takes a lot more of trial and error. So they are not recommended. Anyway, using the method above, we can see that after A makes 12 complete laps and B makes 6 complete laps and three quarters of a complete lap, they will meet for the first time. In other words, after A makes 48 quarter laps and B makes 27 quarter laps, they will meet for the first time.

Let's now reset the speeds the way as follows.
The speed of A is 16 m/s. The speed of B is 13.5 m/s
The lap is still a quarter of a complete lap.

Then, we get 2 equations as follows:
$16t = \pi u$, and $13.5t = \pi v/2$ So we get $u:v = 16:27$.
16 is a multiple of 4, and 27 = 24 + 3, so 27 is a multiple of 4 and 3. Thus, after the particle A makes 16 laps, and the particle B makes 27 laps, they will meet for the first time.
Let's now get the amount of time it takes for them to meet for the first time.
Then, setting first, $u = 16$, we get $16t = \pi u = 16\pi \Rightarrow t = \pi$ sec.
And setting next, $v = 27$, we get $13.5t = \pi v/2 = 27\pi/2 \Rightarrow t = \pi$ sec, too. So after π (approximately 3.14) seconds, the particles will meet for the first time.
What then about the subsequent meetings?

Both particles are at the target point where the tracks meet. So the lap can be now a complete lap. Then, the length of the lap for each particle will be 4 times, since it was a quarter of a complete lap.

So the equations are as follows.

$16t = 2\pi r \times u = 4\pi u$, and $13.5t = 2\pi r \times v = 2\pi v$.

Thus, we get $16t = 4\pi u$, and $27t = 4\pi v$.

So we get $u:v = 16:27$. Thus, the ratio remains the same.

The only difference is the lap length.

So every time A makes 16 complete laps, it meets B, and also, every time B makes 27 complete laps, it meets A.

How about the time it takes?

Setting first, $u = 16$, we get $16t = 4\pi u = 4\pi \times 16$.

So we get $t = 4\pi$. And setting next, $v = 27$, we get

$27t = 4\pi v = 4\pi \times 27$. Thus, we get $t = 4\pi$, too.

Thus, they get to meet every 4π seconds.

Let’s now try some other sets of data.

A starts at the point where the tracks meet, and B starts at the point a quarter track past to the right of the point where the tracks meet. A orbits at 16 m/sec. B orbits at 4.3 m/sec.

And both move clockwise.

The lap is a quarter of a complete lap.

So we get $16t = \pi u$, and $4.3t = \pi v/2 \Rightarrow 8.6t = \pi v$.

Thus, we get $u/v = 16/8.6 = 8/4.3 = 80/43$.

So they will meet for the first time when A makes 80 of its laps and B makes 43 of its laps.

Then, each subsequent meeting will happen every time A makes 80 complete laps and B makes 43 complete laps.

The amount of time it takes for each subsequent meeting will be 4 times the amount of time it takes for them to meet for the first time. You may want to give it a check.

How about the next data?

A starts at the target point where the tracks meet, and B starts at the point a quarter track past to the right of the target point. The speed of A is 16 m/sec. And the speed of B is 4.8 m/sec.

The lap is a quarter of a complete lap.

Then, we get $16t = \pi u$ and $4.8t = \pi v/2 \Rightarrow 9.6t = \pi v$.

So we get $u/v = 16/9.6 = 160/96 = 40/24 = 5/3$.

Thus, they will never make it. The reason is as follows.

If we apply the scale of 4 to each of the entries in the ratio, the number of quarter laps that B makes becomes an even integer, because we get u:v = 20:12.
However, the number of the quarter laps that B makes has to be 3 and a multiple of 4 if it meets A.

Now, we can say that both of the following conditions have to be met at the same time if they meet.

The number of the quarter laps A makes is a multiple of 4, and the number of the quarter laps B makes is 3 and a multiple of 4 if they meet.

In other words, A has to make complete laps, and B has to make complete laps and 3 quarters of a complete lap if they meet.