Examples in Circles and Angles

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0. Put the angles below in radians.

0 $30°$	1. $45°$	2. $60°$	3. $90°$	4. $120°$
5. $180°$	6. $270°$	7. $360°$	8. 405^{o}	9. 775^{o}

(Note that the examples below are for those students who want to take courses in calculus in university level.)
Suppose two vectors are turning in the ***x-y*** plane, and turn about the origin, of course, and both turn in the same direction. So for instance, turning counterclockwise, they keep turning counterclockwise only.

1. Suppose now, one vector is in the second quadrant, and the angle made is θ. What quadrant then, the other vector has to be in if the angle made is $\frac{\theta}{2}$?

2. Suppose again, the angle made by one vector is θ, but this time, the angle made by the other vector is 6θ.

Suppose also, they are in a straight line, and indicate opposite directions, that is, the angle between the two is π. Then,

2.0. find the angle θ.

2.1. Find the angle θ assuming $\mathbf{0} < \theta < \frac{\pi}{2}$.

Suppose this time, they are in a straight line, but indicate the same direction, that is, the angle between the two is 0. Then,

2.2. find the angle θ.

2.3. Find the angle θ assuming $\mathbf{0} < \theta < \frac{\pi}{2}$.

Suppose this time, they are symmetric about the ***y***-axis. Then,

2.4. find the angle θ.

2.5. Find the angle θ assuming $\mathbf{0} < \theta < \frac{\pi}{2}$.

Suppose this time, they are symmetric about the ***x***-axis. Then,

2.6. find the angle θ.

2.7. Find the angle θ assuming $\mathbf{0} < \theta < \frac{\pi}{2}$.

Suggestions or Solutions
To the Problems in the Example 0

To begin with, by definition, we have $180^o = \pi$ rad.
And normally, rad is omitted, so in short, we just set $180^o = \pi$.
And we know 180 degrees is 180 of 1 degrees, that is, 180^o is 180 of 1^os.
In other words, we have $180 \cdot 1^o = \pi$.

So next, putting 1^o in radian, we get $1^o = \dfrac{\pi}{180}$.

And next, we can put an angle $\boldsymbol{A}^o$ this way: $\boldsymbol{A} \cdot 1^o$. And we know $1^o = \dfrac{\pi}{180}$.

So we can get $\boldsymbol{A^o = A \cdot 1^o = A \dfrac{\pi}{180} = \dfrac{A}{180}\pi}$, which is in radians.

That is to say that we get $\boldsymbol{A^o = \dfrac{A}{180}\pi}$. Thus, we can get these:

0. $30° = \dfrac{30}{180}\pi = \dfrac{1}{6}\pi = \dfrac{\pi}{6}$.

1. $45° = \dfrac{45}{180}\pi = \dfrac{1}{4}\pi = \dfrac{\pi}{4}$.

2. $60° = \dfrac{60}{180}\pi = \dfrac{1}{3}\pi = \dfrac{\pi}{3}$.

3. $90° = \dfrac{90}{180}\pi = \dfrac{1}{2}\pi = \dfrac{\pi}{2}$.

4. $120° = 2 \cdot 60° = 2 \cdot \dfrac{\pi}{3} = \dfrac{2\pi}{3}$.

5. $180° = \dfrac{180}{180}\pi = \pi$.

6. $270° = 3 \cdot 90^o = 3 \cdot \dfrac{\pi}{2} = \dfrac{3\pi}{2}$.

7. $360° = \dfrac{360}{180}\pi = 2\pi$.

8. $405^o = 360^o + 45^o = 2\pi + \dfrac{\pi}{4} = \dfrac{9\pi}{4}$.

9. $775^o = \dfrac{775}{180}\pi = \dfrac{155}{36}\pi = \dfrac{155\pi}{36}$, which equals $4\pi + \dfrac{11\pi}{36}$.

Suggestions or Solutions

To the Problem in the Example 1

Two vectors are turning about the origin in the *x*-*y* plane. And both turn in the same direction. One vector is now, in the second quadrant, and the angle made is θ. In what quadrant then the other vector has to be if the angle made is $\frac{\theta}{2}$?

Assuming first, $\frac{\pi}{2} < \boldsymbol{a} < \pi$, and setting $\boldsymbol{\theta} = 2\boldsymbol{n}\pi + \boldsymbol{a}$ where $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$,

we get $\frac{\theta}{2} = \frac{2\boldsymbol{n}\pi + \boldsymbol{a}}{2} = \boldsymbol{n}\pi + \frac{\boldsymbol{a}}{2}$, and $\frac{\pi}{4} < \frac{\boldsymbol{a}}{2} < \frac{\pi}{2}$.

Thus, if $\boldsymbol{n}$ is odd, it is in the third quadrant.
If however, $\boldsymbol{n}$ is even, it is in the first quadrant.

Assuming next, $-\frac{3\pi}{2} < \boldsymbol{a} < -\pi$, and setting $\boldsymbol{\theta} = 2\boldsymbol{n}\pi + \boldsymbol{a}$ where $\boldsymbol{n}$ is an integer $\leq \mathbf{0}$,

we get $\frac{\theta}{2} = \frac{2\boldsymbol{n}\pi + \boldsymbol{a}}{2} = \boldsymbol{n}\pi + \frac{\boldsymbol{a}}{2}$, and $-\frac{3\pi}{4} < \frac{\boldsymbol{a}}{2} < -\frac{\pi}{2}$.

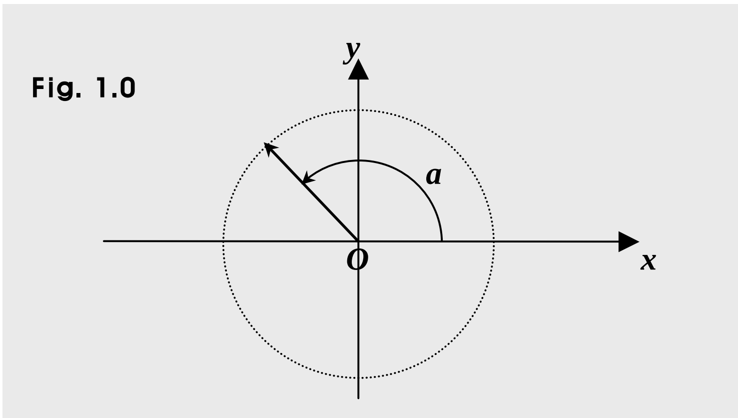
Thus, if $\boldsymbol{n}$ is odd, it is in the first quadrant.
If however, $\boldsymbol{n}$ is even, it is in the third quadrant.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, can we just set $\frac{\pi}{2} < \boldsymbol{\theta} < \pi$, since the vector is in the second quadrant?
We don't know how many complete turns the vector in the second quadrant has made before it made the angle $\boldsymbol{\theta}$. Also, we don't know in what direction it turns, that is, we don't know if it turns counterclockwise or not. How then can we set the angle $\boldsymbol{\theta}$?

We can have two cases where the vectors turn counterclockwise, and they turn clockwise. So suppose first, the vectors turn counterclockwise.
Then first, making a complete turn, the angle made by the vector is 2π.

Next, if the vector is in the second quadrant, the angle between the vector and the ***x***-axis on the right of the origin is between $\frac{\pi}{2}$ and π.



So assuming $\frac{\pi}{2} < \boldsymbol{a} < \pi$, we want to set $\boldsymbol{\theta = 2n\pi + a}$ where $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$.
How then can we set the other angle made by the other vector?

The angle made is $\frac{\theta}{2}$, and we have $\boldsymbol{\theta = 2n\pi + a}$.

So $\frac{\theta}{2} = \frac{2\boldsymbol{n}\pi + \boldsymbol{a}}{2} = \boldsymbol{n}\pi + \frac{\boldsymbol{a}}{2}$. And we know $\frac{\pi}{2} < \boldsymbol{a} < \pi$. So we get $\frac{\pi}{4} < \frac{\boldsymbol{a}}{2} < \frac{\pi}{2}$.

That is to say that $\frac{\boldsymbol{a}}{2}$ is an acute angle.

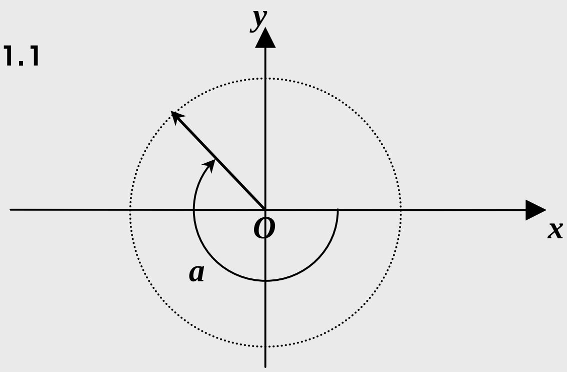
In what quadrant then is the other vector if $\boldsymbol{n}$ is odd?

If $\boldsymbol{n}$ is odd, so if for instance, $\boldsymbol{n} = \mathbf{1}$, we get $\frac{\theta}{2} = \pi + \frac{\boldsymbol{a}}{2}$, so it's in the third quadrant, and if $\boldsymbol{n} = \mathbf{3}$, we get $\frac{\theta}{2} = \mathbf{3}\pi + \frac{\boldsymbol{a}}{2}$, so it's in the third quadrant again. If however, $\boldsymbol{n}$ is not odd, so for instance, if $\boldsymbol{n} = \mathbf{0}$, we get $\frac{\theta}{2} = \frac{\boldsymbol{a}}{2}$, so it's in the first quadrant, and if $\boldsymbol{n} = \mathbf{2}$ or $\mathbf{4}$, we get $\frac{\theta}{2} = \mathbf{2}\pi + \frac{\boldsymbol{a}}{2}$, or $\frac{\theta}{2} = \mathbf{4}\pi + \frac{\boldsymbol{a}}{2}$, so it's in the first quadrant again.

Thus, if $\boldsymbol{n}$ is odd, it is in the third quadrant. And if $\boldsymbol{n}$ is not odd, it is in the first quadrant.

Suppose next, the vectors turn clockwise.
Then first, making a complete turn, the angle made by the vector is -2π.
Next, if the vector is in the second quadrant, the angle between the vector and the $\boldsymbol{x}$-axis on the right of the origin is between $-\frac{3\pi}{2}$ and $-\pi$.

Fig. 1.1



So assuming $-\frac{3\pi}{2} < \boldsymbol{a} < -\pi$, we want to set $\boldsymbol{\theta} = 2\boldsymbol{n}\pi + \boldsymbol{a}$ where $\boldsymbol{n}$ is an integer $\leq \mathbf{0}$.

How then can we set the other angle made by the other vector?

The angle made is $\frac{\theta}{2}$, and we have $\boldsymbol{\theta} = 2\boldsymbol{n}\pi + \boldsymbol{a}$. So $\frac{\theta}{2} = \frac{2\boldsymbol{n}\pi + \boldsymbol{a}}{2} = \boldsymbol{n}\pi + \frac{\boldsymbol{a}}{2}$.

And we know $-\frac{3\pi}{2} < \boldsymbol{a} < -\pi$. So we get $-\frac{3\pi}{4} < \frac{\boldsymbol{a}}{2} < -\frac{\pi}{2}$.

That is to say that $\frac{\boldsymbol{a}}{2}$ is an obtuse angle negative.

In what quadrant then is the other vector if $\boldsymbol{n}$ is odd?

If $\boldsymbol{n}$ is odd, so for instance, if $\boldsymbol{n} = -1$, we get $\frac{\theta}{2} = -\pi + \frac{\boldsymbol{a}}{2}$, so it's in the first quadrant, and if $\boldsymbol{n} = -3$, we get $\frac{\theta}{2} = -3\pi + \frac{\boldsymbol{a}}{2}$, so it's in the first quadrant again. If however, $\boldsymbol{n}$ is not odd, so for instance, if $\boldsymbol{n} = \mathbf{0}$, we get $\frac{\theta}{2} = \frac{\boldsymbol{a}}{2}$, so it's in the third quadrant, and if $\boldsymbol{n} = -2$ or -4, we get $\frac{\theta}{2} = -2\pi + \frac{\boldsymbol{a}}{2}$, or $\frac{\theta}{2} = -4\pi + \frac{\boldsymbol{a}}{2}$, so it's in the third quadrant again.
Thus, if $\boldsymbol{n}$ is odd, it is in the first quadrant. And if $\boldsymbol{n}$ is not odd, it is in the third quadrant.

Suggestions or Solutions
To the Problems 0 and 1 in the Example 2

Two vectors are turning about the origin in the *x-y* plane. And both turn in the same direction. Suppose now, the angle made by one vector is θ, the angle made by the other vector is 6θ, and they are in a straight line, and indicate opposite directions, that is, the angle between the two is π. Then,

2.0. find the angle θ. **2.1. Find the angle θ assuming $0 < \theta < \pi/2$.**

Assuming first, the vectors turn counterclockwise, and thus, setting $\mathbf{6\theta - \theta = 2n\pi + \pi}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$, we get $\mathbf{5\theta = (2n + 1)\pi} \Rightarrow \theta = \dfrac{(2\boldsymbol{n}+1)\pi}{5}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

And assuming next, the vectors turn clockwise, and thus, setting $\mathbf{\theta - 6\theta = 2n\pi + \pi}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$, we get $\mathbf{-5\theta = (2n + 1)\pi} \Rightarrow \theta = -\dfrac{(2\boldsymbol{n}+1)\pi}{5}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

Next, assuming $\mathbf{0} < \theta < \pi/\mathbf{2}$, we want to use $\theta = \dfrac{(2\boldsymbol{n}+1)\pi}{5}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$, since θ is positive, so we get $\mathbf{0} < \dfrac{(2\boldsymbol{n}+1)\pi}{5} < \pi/\mathbf{2} \Rightarrow \mathbf{0 < (2n + 1)\pi < 5\pi/2} \Rightarrow \mathbf{0 < 2n + 1 < 5/2}$ $\Rightarrow \mathbf{-1 < 2n < 3/2} \Rightarrow \mathbf{-1/2 < n < 3/4} \Rightarrow \mathbf{n = 0}$ since $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$.

Thus, the angle θ is $\pi/\mathbf{5}$ if $\mathbf{0} < \theta < \pi/\mathbf{2}$.

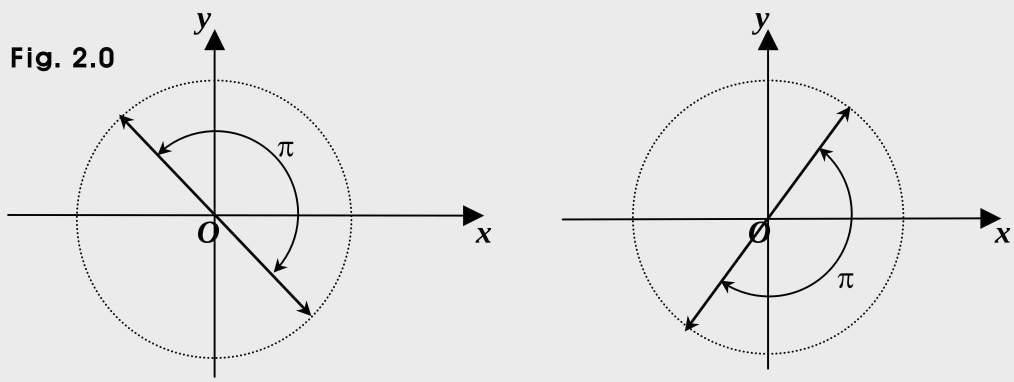
If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, can we just set $\mathbf{6\theta - \theta = \pi}$, since the angle between the two vectors is π?

We don't know how many complete turns one vector has made before it made the angle θ. Also, we don't know how many complete turns the other vector has made before it made the angle $\mathbf{6\theta}$. How then can we set the difference between the two angles made?

Unlike the previous example, the direction of turning does not really matter.
What matters is the difference between the two angles made, and it is π.

Fig. 2.0



And we don't know how many complete turns each vector made before each angle is made. What we know is though, the fact that making a complete turn, the angle made by each vector is 2π.
So assuming the vectors turn counterclockwise, we can set the difference the way as follows. $\boldsymbol{6\theta - \theta = 2n\pi + \pi}$ for $\boldsymbol{n}$ integer $\geq$ **0**.

Thus, we get $\boldsymbol{5\theta = (2n + 1)\pi \Rightarrow \theta = \frac{(2n+1)\pi}{5}}$ for $\boldsymbol{n}$ integer $\geq$ **0**.

And next, assuming the vectors turn clockwise, we can set the difference the way as follows. $\boldsymbol{\theta - 6\theta = 2n\pi + \pi}$ for $\boldsymbol{n}$ integer $\geq$ **0**.

Then, we get $\boldsymbol{-5\theta = (2n + 1)\pi \Rightarrow \theta = -\frac{(2n+1)\pi}{5}}$ for $\boldsymbol{n}$ integer $\geq$ **0**.

And next, we want to find the angle $\boldsymbol{\theta}$ assuming $\boldsymbol{0 < \theta < \pi/2}$.

Then, the angle $\boldsymbol{\theta}$ is positive, so we want to consider the case where the vectors turn counterclockwise.

We have $\boldsymbol{\theta = \frac{(2n+1)\pi}{5}}$, and $\boldsymbol{0 < \theta < \pi/2}$.

So we get $\boldsymbol{0 < \frac{(2n+1)\pi}{5} < \pi/2 \Rightarrow 0 < (2n + 1)\pi < 5\pi/2 \Rightarrow 0 < 2n + 1 < 5/2}$

$\boldsymbol{\Rightarrow -1 < 2n < 3/2 \Rightarrow -1/2 < n < 3/4 \Rightarrow n = 0}$ since $\boldsymbol{n}$ is an integer $\geq$ **0**.

Thus, the angle $\boldsymbol{\theta}$ is $\boldsymbol{\pi/5}$ if $\boldsymbol{0 < \theta < \pi/2}$.

Suggestions or Solutions
To the Problems 2 and 3 in the Example 2

Two vectors are turning about the origin in the *x-y* plane. And both turn in the same direction. Suppose now, the angle made by one vector is θ, the angle made by the other vector is 6θ, and they are in a straight line, and indicate the same directions, that is, the angle between the two is 0. Then,

2.2. find the angle θ. **2.3. Find the angle θ assuming $0 < \theta < \pi/2$.**

Assuming first, the vectors turn counterclockwise, and thus, setting $\mathbf{6\theta - \theta = 2n\pi}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$, we get $\mathbf{5\theta = 2n\pi} \Rightarrow \theta = \dfrac{2\boldsymbol{n}\pi}{5}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

Assuming next, the vectors turn clockwise; thus, setting $\mathbf{\theta - 6\theta = 2n\pi}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$, we get $\mathbf{-5\theta = 2n\pi} \Rightarrow \theta = -\dfrac{2\boldsymbol{n}\pi}{5}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

And next, assuming $\mathbf{0} < \theta < \pi/\mathbf{2}$, we want to use $\theta = \mathbf{2n\pi/5}$, since the angle θ is positive, so we get $\mathbf{0} < \dfrac{2\boldsymbol{n}\pi}{5} < \pi/\mathbf{2} \Rightarrow \mathbf{0 < 2n\pi < 5\pi/2} \Rightarrow \mathbf{0 < 2n < 5/2} \Rightarrow \mathbf{0 < n < 5/4}$

$\Rightarrow \boldsymbol{n} = \mathbf{1}$ since $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$.

Thus, the angle θ is $\mathbf{2\pi/5}$ if $\mathbf{0} < \theta < \pi/\mathbf{2}$.

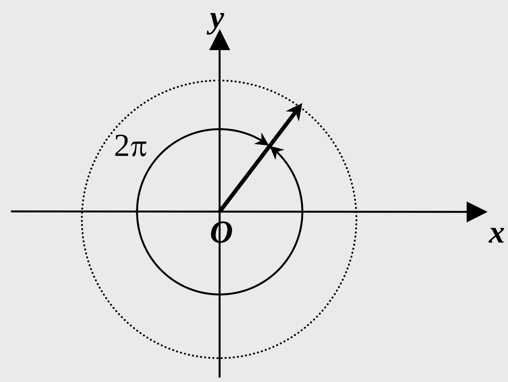
If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, can we just set $\mathbf{6\theta - \theta = 0}$, since the angle between the two vectors is 0?

We don’t know how many complete turns one vector has made before it made the angle θ. Also, we don’t know how many complete turns the other vector has made before it made the angle $\mathbf{6\theta}$. How then can we set the difference between the two angles made?

As in the case of the previous example, what matters is the difference between the two angles made, and it is 0.

Fig. 2.1



And since the angle between the two is 0, we can also say that the angle between the two is 2π. And we don't know how many complete turns each vector made before each angle is made. What we know is though, the fact that making a complete turn, the angle made by each vector is 2π.

So assuming the vectors turn counterclockwise, we can set the difference the way as follows. $\mathbf{6\theta - \theta = 2n\pi}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

Thus, we get $\mathbf{5\theta = 2n\pi} \Rightarrow \theta = \dfrac{2n\pi}{5}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

And next, assuming the vectors turn clockwise, we can set the difference the way as follows. $\mathbf{\theta - 6\theta = 2n\pi}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

Then, we get $\mathbf{-5\theta = 2n\pi} \Rightarrow \theta = -\dfrac{2n\pi}{5}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

And next, we want to find the angle θ assuming $\mathbf{0 < \theta < \pi/2}$.

Then, the angle θ is positive, so we want to consider the case where the vectors turn counterclockwise.

We have $\theta = \dfrac{2n\pi}{5}$, and $\mathbf{0 < \theta < \pi/2}$.

So we get $\mathbf{0} < \dfrac{2n\pi}{5} < \mathbf{\pi/2} \Rightarrow \mathbf{0 < 2n\pi < 5\pi/2} \Rightarrow \mathbf{0 < 2n < 5/2} \Rightarrow \mathbf{0 < n < 5/4} \Rightarrow \mathbf{n = 1}$, since $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$.

Thus, the angle θ is $\mathbf{2\pi/5}$ if $\mathbf{0 < \theta < \pi/2}$.

Suggestions or Solutions
To the Problems 4 and 5 in the Example 2

Two vectors are turning about the origin in the *x*-*y* plane. And both turn in the same direction. Suppose now, the angle made by one vector is θ, the angle made by the other vector is 6θ, and they are symmetric about the *y*-axis. Then,

2.4. find the angle θ. **2.5. Find the angle θ assuming $0 < \theta < \pi/2$.**

Assuming first, the vectors turn counterclockwise, and n is an integer ≥ 0,
we can set $\theta + 6\theta = 2n\pi + \pi$ or $2n\pi + 3\pi$, since the angles are positive.

Meanwhile, $2n\pi + \pi = (2n + 1)\pi$,
and $2n\pi + 3\pi = 2n\pi + 2\pi + \pi = 2(n + 1)\pi + \pi = \{2(n + 1) + 1\}\pi$.

And we know $2n + 1$ is an integer odd, and $\{2(n + 1) + 1\}$ is an integer odd, too.
So we can set $\theta + 6\theta = (2n + 1)\pi$ for n integer ≥ 0.
Thus, we get $7\theta = (2n + 1)\pi \Rightarrow \theta = (2n + 1)\pi/7$.

Assuming next, the vectors turn clockwise, since the angles are negative,
we can set $\theta + 6\theta = -(2n\pi + \pi)$ or $-(2n\pi + 3\pi)$.

Meanwhile, $-(2n\pi + \pi) = -(2n + 1)\pi$, and $-(2n\pi + 3\pi) = -(2n\pi + 2\pi + \pi)$
$= -\{2(n + 1)\pi + \pi\} = -\{2(n + 1) + 1\}\pi$.

And we know $2n + 1$ is an integer odd, and so is $\{2(n + 1) + 1\}$.
So we can set $\theta + 6\theta = -(2n + 1)\pi$ for n integer ≥ 0.
Thus, we get $7\theta = -(2n + 1)\pi \Rightarrow \theta = -\dfrac{(2n+1)\pi}{7}$.

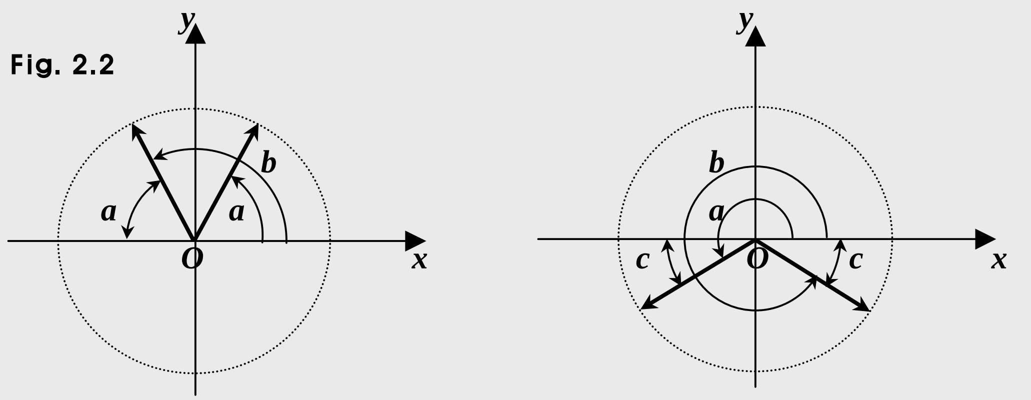
Next, assuming $0 < \theta < \pi/2$, we use $\theta = \dfrac{(2n+1)\pi}{7}$, since the angle θ is positive,

so we get $0 < \dfrac{(2n+1)\pi}{7} < \pi/2 \Rightarrow 0 < (2n + 1)\pi < 7\pi/2 \Rightarrow 0 < 2n + 1 < 7/2$
$\Rightarrow -1 < 2n < 5/2 \Rightarrow -1/2 < n < 5/4 \Rightarrow n = 0$ or 1, since n is an integer ≥ 0.
Thus, the angle θ is $\pi/7$ or $3\pi/7$ if $0 < \theta < \pi/2$.

If not quite sure of the idea behind the processes above, follow the steps below.

We can have two cases where the two vectors are symmetric about the $\boldsymbol{y}$-axis.

Fig. 2.2



Let's begin with the case where the vectors turn counterclockwise.
Then first, in the graph on the left, we can see that $\boldsymbol{a + b = \pi}$.

Next, in the graph on the right, we can see that $\boldsymbol{a = \pi + c}$, and $\boldsymbol{b = 2\pi - c}$, and thus, we get $\boldsymbol{a + b = (\pi + c) + (2\pi - c) = 3\pi}$.

And we don't know how many complete turns each of the vectors has made.
So taking the sum of the two angles $\boldsymbol{\theta}$ and $\boldsymbol{6\theta}$, and assuming $\boldsymbol{n}$ is an integer $\geq \boldsymbol{0}$, we can put it the way as follows: $\boldsymbol{\theta + 6\theta = 2n\pi + \pi}$ or $\boldsymbol{2n\pi + 3\pi}$.

However, we can put the sum the way below.
$\boldsymbol{2n\pi + \pi = (2n + 1)\pi}$, and $\boldsymbol{2n\pi + 3\pi = 2n\pi + 2\pi + \pi = 2(n + 1)\pi + \pi = \{2(n + 1) + 1\}\pi}$.

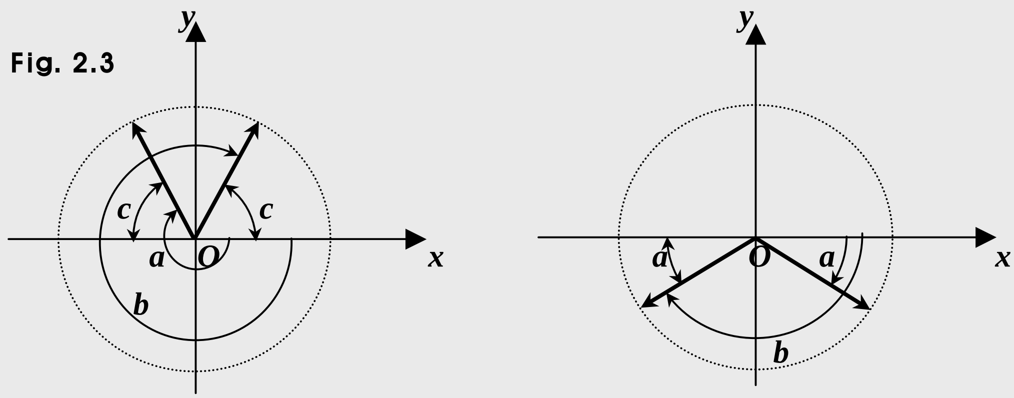
Thus, we get $\boldsymbol{\theta + 6\theta = (2n + 1)\pi}$ or $\boldsymbol{\{2(n + 1) + 1\}\pi}$. So what?
We know $\boldsymbol{2n + 1}$ is an integer odd, and $\boldsymbol{\{2(n + 1) + 1\}}$ is an integer odd, too.

So we can simply put the sum the way as follows: $\boldsymbol{\theta + 6\theta = (2n + 1)\pi}$ for $\boldsymbol{n}$ integer $\geq \boldsymbol{0}$.

Thus, we get $\boldsymbol{7\theta = (2n + 1)\pi \Rightarrow \theta} = \dfrac{(2\boldsymbol{n}+1)\pi}{7}$.

And let's next, move on to the case where the vectors turn clockwise.
Then, to begin with, we can put some angles the way as follows.

Fig. 2.3



Then first, in the graph on the right, we can see that $\boldsymbol{a} + \boldsymbol{b} = -\pi$.

Next, in the graph on the left, we can see that $\boldsymbol{a} = -(\pi + \boldsymbol{c})$, and $\boldsymbol{b} = -(2\pi - \boldsymbol{c})$, and thus, we get $\boldsymbol{a} + \boldsymbol{b} = -(\pi + \boldsymbol{c}) - (2\pi - \boldsymbol{c}) = -3\pi$.

And we don't know how many complete turns each of the vectors has made.
So taking the sum of the two angles θ and 6θ, and assuming $\boldsymbol{n}$ is an integer ≥ 0, we can put it the way as follows: $\theta + 6\theta = -(2n\pi + \pi)$ or $-(2n\pi + 3\pi)$.

And we can put the sum the way below.
$-(2n\pi + \pi) = -(2n + 1)\pi$, and $-(2n\pi + 3\pi) = -(2n\pi + 2\pi + \pi)$
$= -\{2(n + 1)\pi + \pi\} = -\{2(n + 1) + 1\}\pi$.

Thus, we get $\theta + 6\theta = -(2n + 1)\pi$ or $-\{2(n + 1) + 1\}\pi$.

And we know $2n + 1$ is an integer odd, and $\{2(n + 1) + 1\}$ is an integer odd, too.

So we can simply put the sum the way as follows: $\theta + 6\theta = -(2n + 1)\pi$ for $\boldsymbol{n}$ integer ≥ 0.

Thus, we get $7\theta = -(2n + 1)\pi \Rightarrow \theta = -\dfrac{(2\boldsymbol{n}+1)\pi}{7}$.

And next, we want to find the angle θ assuming $0 < \theta < \pi/2$.

Then, the angle θ is positive, so we want to consider the case where the vectors turn counterclockwise.

We have $\theta = (2n + 1)\pi/7$, and $0 < \theta < \pi/2$.

So we get $0 < \dfrac{(2n+1)\pi}{7} < \pi/2 \Rightarrow 0 < (2n + 1)\pi < 7\pi/2 \Rightarrow 0 < 2n + 1 < 7/2$

$\Rightarrow -1 < 2n < 5/2 \Rightarrow -1/2 < n < 5/4 \Rightarrow n = 0$ or 1 since n is an integer ≥ 0.

Thus, the angle θ is $\pi/7$ or $3\pi/7$ if $0 < \theta < \pi/2$.

In short:

Assuming first, the vectors turn counterclockwise, and n is an integer ≥ 0, we can set $\theta + 6\theta = 2n\pi + \pi$ or $2n\pi + 3\pi$. Meanwhile, $2n\pi + \pi = (2n + 1)\pi$,

and $2n\pi + 3\pi = 2n\pi + 2\pi + \pi = 2(n + 1)\pi + \pi = \{2(n + 1) + 1\}\pi$.

And we know $2n + 1$ is an integer odd, and $\{2(n + 1) + 1\}$ is an integer odd, too.

So we can set $\theta + 6\theta = (2n + 1)\pi$ for n integer ≥ 0.

Thus, we get $7\theta = (2n + 1)\pi \Rightarrow \theta = \dfrac{(2n+1)\pi}{7}$.

Assuming next, the vectors turn clockwise,

we can set $\theta + 6\theta = -(2n\pi + \pi)$ or $-(2n\pi + 3\pi)$.

Meanwhile, $-(2n\pi + \pi) = -(2n + 1)\pi$, and $-(2n\pi + 3\pi) = -(2n\pi + 2\pi + \pi)$

$= -\{2(n + 1)\pi + \pi\} = -\{2(n + 1) + 1\}\pi$.

And we know $2n + 1$ is an integer odd, and so is $\{2(n + 1) + 1\}$.

So we can set $\theta + 6\theta = -(2n + 1)\pi$ for n integer ≥ 0.

Thus, we get $7\theta = -(2n + 1)\pi \Rightarrow \theta = -\dfrac{(2n+1)\pi}{7}$.

And next, assuming $0 < \theta < \pi/2$, we want to use $\theta = \dfrac{(2n+1)\pi}{7}$, since the angle θ is positive, so we get $0 < (2n + 1)\pi/7 < \pi/2 \Rightarrow 0 < (2n + 1)\pi < 7\pi/2 \Rightarrow 0 < 2n + 1 < 7/2$

$\Rightarrow -1 < 2n < 5/2 \Rightarrow -1/2 < n < 5/4 \Rightarrow n = 0$ or 1 since n is an integer ≥ 0.

Thus, the angle θ is $\pi/7$ or $3\pi/7$ if $0 < \theta < \pi/2$.

Suggestions or Solutions
To the Problems 6 and 7 in the Example 2

Suppose two vectors are turning in the *x-y* plane, and they turn about the origin, of course. Suppose now, the angle made by one vector is θ, the angle made by the other vector is 6θ, and they are symmetric about the *x*-axis. Then,

2.6. find the angle θ. **2.7. Find the angle θ assuming $0 < \theta < \pi/2$.**

Assuming first, the vectors turn counterclockwise, and n is an integer ≥ 0,
we can set $\theta + 6\theta = 2n\pi + 2\pi$.

And we know $2n$ is an integer even, and so is 2.

So we get $7\theta = 2n\pi \Rightarrow \theta = \frac{2n\pi}{7}$ for n integer ≥ 0.

Assuming next, the vectors turn clockwise, we can set $\theta + 6\theta = -(2n\pi + 2\pi)$.
And we know $2n$ is an integer even, and so is 2.

So we get $7\theta = -2n\pi \Rightarrow \theta = -\frac{2n\pi}{7}$ for n integer ≥ 0.

Thus, putting threads together, we get $\theta = \frac{2n\pi}{7}$ for n integer.

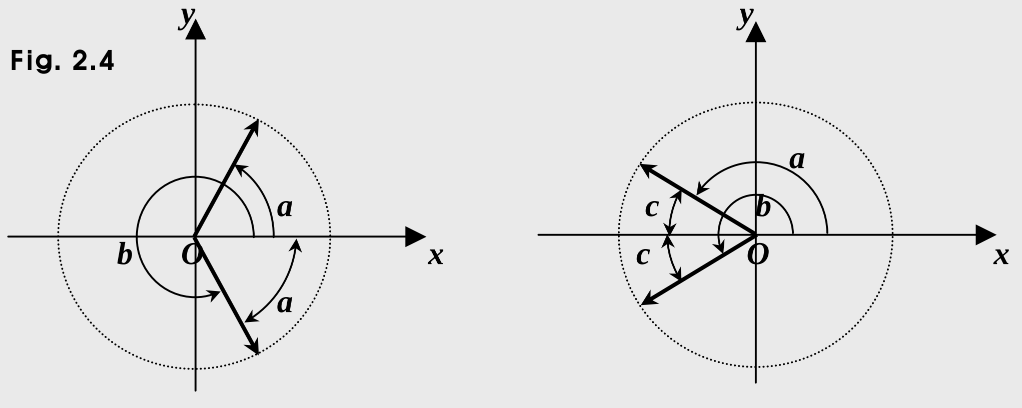
Next, assuming $0 < \theta < \pi/2$, we want to use $\theta = \frac{2n\pi}{7}$ for n integer ≥ 0, since the angle θ is positive, so we get $0 < \frac{2n\pi}{7} < \pi/2 \Rightarrow 0 < 2n\pi < 7\pi/2 \Rightarrow 0 < 2n < 7/2$
$\Rightarrow 0 < n < 7/4 \Rightarrow n = 1$ since n is an integer ≥ 0.

Thus, the angle θ is $2\pi/7$ if $0 < \theta < \pi/2$.

If not quite sure of the idea behind the processes above, follow the steps below.

As in the case of the previous example, we can have two cases where the two vectors are symmetric about the ***x***-axis.

Fig. 2.4



Let's begin with the case where the vectors turn counterclockwise.
Then first, in the graph on the left, we can see that $\boldsymbol{b} = 2\pi - \boldsymbol{a}$, so we get $\boldsymbol{a} + \boldsymbol{b} = 2\pi$.

Next, in the graph on the right, we can see that $\boldsymbol{a} = \pi - \boldsymbol{c}$, and $\boldsymbol{b} = \pi + \boldsymbol{c}$, and thus, we get $\boldsymbol{a} + \boldsymbol{b} = (\pi - \boldsymbol{c}) + (\pi + \boldsymbol{c}) = 2\pi$.

And we don't know how many complete turns each of the vectors has made.

So taking the sum of the two angles $\boldsymbol{\theta}$ and $\mathbf{6}\boldsymbol{\theta}$, and assuming $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$, we can put it the way as follows: $\boldsymbol{\theta} + 6\boldsymbol{\theta} = 2\boldsymbol{n}\pi + 2\pi$.

And we know: $\mathbf{2}\boldsymbol{n}$ is an integer even, and $\mathbf{2}$ is an integer even, too.

So we can simply put the sum the way as follows: $\boldsymbol{\theta} + 6\boldsymbol{\theta} = 2\boldsymbol{n}\pi$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

Thus, we get $7\boldsymbol{\theta} = 2\boldsymbol{n}\pi \Rightarrow \boldsymbol{\theta} = \dfrac{2\boldsymbol{n}\pi}{7}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$.

And let's next, move on to the case where the vectors turn clockwise.
Then, to begin with, we can put some angles the way below.

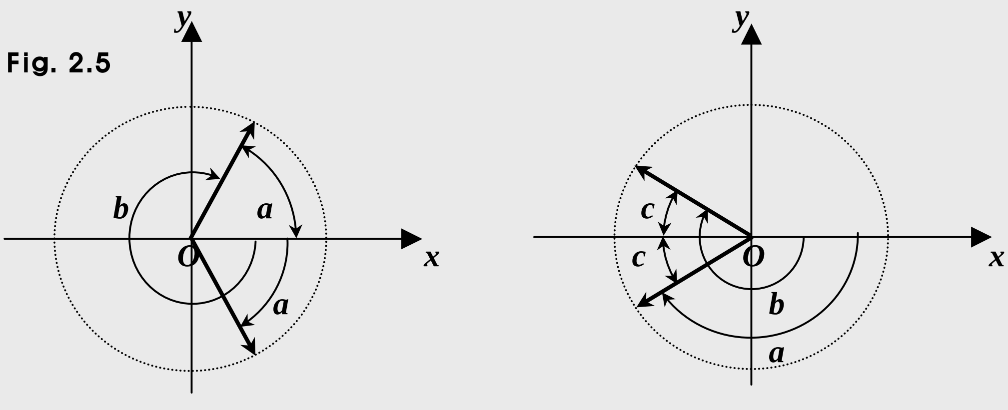


Fig. 2.5

Then first, in the graph on the right, we can see that $\boldsymbol{a + b = -2\pi}$.
Next, in the graph on the left, assuming $\boldsymbol{c}$ is an angle positive, we can see this:

$\boldsymbol{a = -\pi + c}$, and $\boldsymbol{b = -\pi - c}$, and thus, we get $\boldsymbol{a + b = -\pi + c - \pi - c = -2\pi}$.

And we don't know how many complete turns each of the vectors has made.
So taking the sum of the two angles $\boldsymbol{\theta}$ and $\boldsymbol{6\theta}$, and assuming $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$, we can put it the way as follows: $\boldsymbol{\theta + 6\theta = -2n\pi - 2\pi = -(2n\pi + 2\pi)}$.

And we know $\boldsymbol{2n}$ is an integer even, and so is $\mathbf{2}$, of course.
So we can simply put the sum the way as follows: $\boldsymbol{\theta + 6\theta = 2n\pi}$ for $\boldsymbol{n}$ integer $\leq \mathbf{0}$.

Thus, we get $\boldsymbol{7\theta = 2n\pi} \Rightarrow \theta = \dfrac{2\boldsymbol{n}\pi}{7}$ for $\boldsymbol{n}$ integer $\leq \mathbf{0}$.

Now, putting threads together, we have

$\theta = \dfrac{2\boldsymbol{n}\pi}{7}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$ if the vectors turn counterclockwise.

$\theta = \dfrac{2\boldsymbol{n}\pi}{7}$ for $\boldsymbol{n}$ integer $\leq \mathbf{0}$ if the vectors turn clockwise.

So either way, we can put it this way: $\theta = \dfrac{2\boldsymbol{n}\pi}{7}$ where $\boldsymbol{n}$ is an integer.

That's because the statement that $\boldsymbol{n}$ is an integer means that $\boldsymbol{n}$ can be positive, 0, or negative.

And if $n \geq 0$, the vectors turn counterclockwise, and if $n < 0$, the vectors turn clockwise. What if $n = 0$ though?

If $n = 0$, we get $\theta = 0$, and thus, $6\theta = 0$, too.

In that case, the two vectors both are on the x-axis, which means technically, the two vectors are symmetric about the x-axis.

Next, we want to find the angle θ assuming $0 < \theta < \pi/2$.
Then, the angle θ is positive, so we want to consider the case where the vectors turn counterclockwise.

We have $\theta = \frac{2n\pi}{7}$, and $0 < \theta < \pi/2$. So we get

$0 < \frac{2n\pi}{7} < \pi/2 \Rightarrow 0 < 2n\pi < 7\pi/2 \Rightarrow 0 < 2n < 7/2 \Rightarrow 0 < n < 7/4$

$\Rightarrow n = 1$ since n is an integer ≥ 0.

Thus, the angle θ is $2\pi/7$ if $0 < \theta < \pi/2$.

In short:

Assuming first, the vectors turn counterclockwise, and n is an integer ≥ 0,
we can set $\theta + 6\theta = 2n\pi + 2\pi$.
And we know $2n$ is an integer even, and so is 2.
So we get $7\theta = 2n\pi \Rightarrow \theta = \frac{2n\pi}{7}$ for n integer ≥ 0.
Assuming next, the vectors turn clockwise, we can set $\theta + 6\theta = -(2n\pi + 2\pi)$.

And we know $2n$ is an integer even, and so is 2.

So we get $7\theta = -2n\pi \Rightarrow \theta = -\frac{2n\pi}{7}$ for n integer ≥ 0.

Thus, putting threads together, we get $\boldsymbol{\theta} = \frac{2\boldsymbol{n}\pi}{7}$ for $\boldsymbol{n}$ integer.

And next, assuming $\mathbf{0} < \boldsymbol{\theta} < \boldsymbol{\pi/2}$, we want to use $\boldsymbol{\theta} = \frac{2\boldsymbol{n}\pi}{7}$ for $\boldsymbol{n}$ integer $\geq \mathbf{0}$, since the angle $\boldsymbol{\theta}$ is positive, so we get $\mathbf{0} < \frac{2\boldsymbol{n}\pi}{7} < \boldsymbol{\pi/2} \Rightarrow \mathbf{0 < 2}\boldsymbol{n}\boldsymbol{\pi} < \mathbf{7}\boldsymbol{\pi}\mathbf{/2} \Rightarrow \mathbf{0 < 2}\boldsymbol{n} < \mathbf{7/2}$
$\Rightarrow \mathbf{0} < \boldsymbol{n} < \mathbf{7/4} \Rightarrow \boldsymbol{n} = \mathbf{1}$ since $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$.

Thus, the angle $\boldsymbol{\theta}$ is $\mathbf{2}\boldsymbol{\pi}\mathbf{/7}$ if $\mathbf{0} < \boldsymbol{\theta} < \boldsymbol{\pi/2}$.