An example in Triangles and Trigonometry

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An Example in Triangles and Trigonometry

Find the value of the cosine of 15^{o}, that is, $\mathbf{cos\ 15^{o}}$, algebraically, of course.

By the way, if an angle is very small, and is in radian, we can get each trig-ratio of the angle the way below, too.

(Note that the formulas below are not covered in high school math.)

$$\mathbf{\sin x = \sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n+1}}{(2n+1)!}}$$

$$\mathbf{\cos x = \sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n}}{2n!}}$$

$$\mathbf{\tan x} = \frac{\sin x}{\cos x}$$

For instance, $\mathbf{\sin 0.1 = \sum_{n=0}^{\infty}(-1)^{n}\frac{0.1^{2n+1}}{(2n+1)!}}$

$$= \frac{\{(-1)^{0}(0.1)^{1}\}}{1!} + \frac{\{(-1)^{1}(0.1)^{3}\}}{3!} + \frac{\{(-1)^{2}(0.1)^{5}\}}{5!} + \frac{\{(-1)^{3}(0.1)^{7}\}}{7!} + \dots$$

$$= \frac{1\cdot 0.1}{1} - \frac{0.1^{3}}{1\cdot 2\cdot 3} + \frac{0.1^{5}}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{0.1^{7}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \dots$$

$$= \frac{0.1}{1} - \frac{0.1^{3}}{6} + \frac{0.1^{5}}{120} - \frac{0.1^{7}}{720} + \dots$$

$$= 0.1 - \frac{0.001}{6} + \frac{0.00001}{120} - \frac{0.0000001}{720} + \dots$$

$= 0.1 - 0.0001666\dots + 0.000000083333\dots - 0.000000000138888\dots + \dots$

≈ 0.0998334165

And using a calculator, we can get **sin 0.1 =** 0.09983341664682815.

Suggestions or Solutions
To the Example in Triangles and Trigonometry

Find the value of the cosine of 15°, that is, cos 15°.

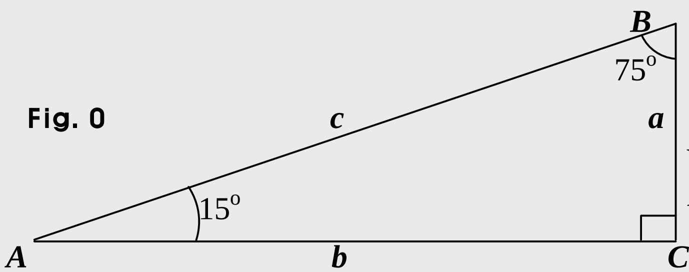
We have $\mathbf{cos\,30^{o}} = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}$, $\mathbf{cos\,45^{o}} = \dfrac{\sqrt{\mathbf{2}}}{\mathbf{2}}$, and $\mathbf{cos\,60^{o}} = \dfrac{\mathbf{1}}{\mathbf{2}}$.

And getting the ratios above, we use two right triangles, one is half a regular triangle, and the other is a right triangle isosceles.
How then can we get the ratio, **cos** 15°?

We know 15° is the angle between the adjacent and the hypotenuse.   So?

So using a right triangle where an angle is 15°, we should be able to get the value of the trig ratio, **cos** 15°. Thus, forming such a triangle, we can get this:

Fig. 0



We know that the cosine is the adjacent over the hypotenuse.

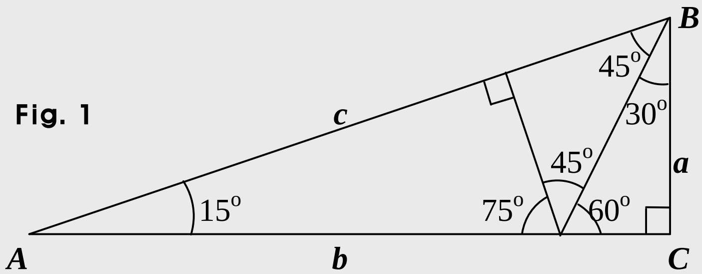
So we can set $\mathbf{cos\,15^{o}} = \dfrac{\boldsymbol{b}}{\boldsymbol{c}}$. But we don't know the values of ***b*** and ***c***.

We know however $\mathbf{cos\,30^{o}} = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}$, $\mathbf{cos\,45^{o}} = \dfrac{\sqrt{\mathbf{2}}}{\mathbf{2}}$, and $\mathbf{cos\,60^{o}} = \dfrac{\mathbf{1}}{\mathbf{2}}$.

So we may want to take advantage of those ratios.   How though?

We can notice that we can form inside the triangle above a right triangle where two of the angles are 30° and 60°. Thus, forming it, we can get this:

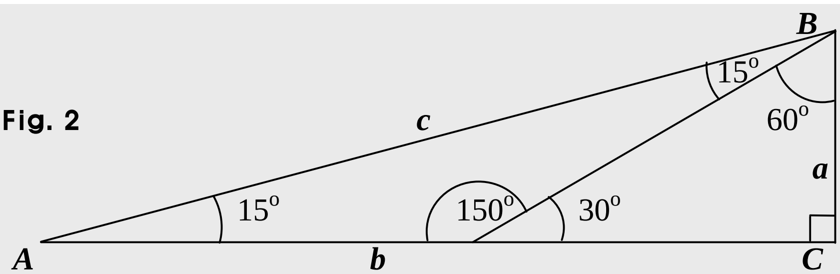
Fig. 1



We don't really get anything useful though.

So we may want to try again, forming another right triangle, where two of the angles are 30° and 60°, of course.
Thus, forming it a bit differently, we can get the one below.

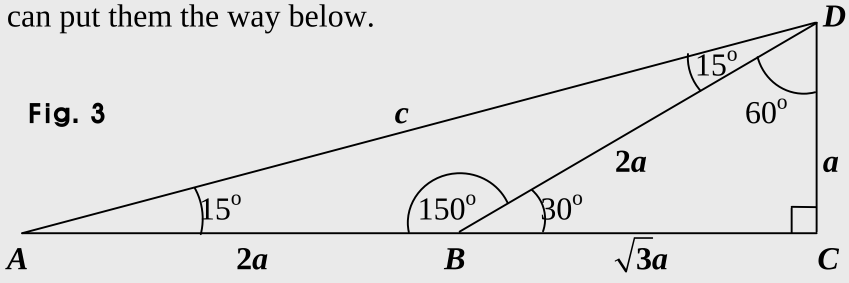
Fig. 2



Now, we can see something useful. What then is it?

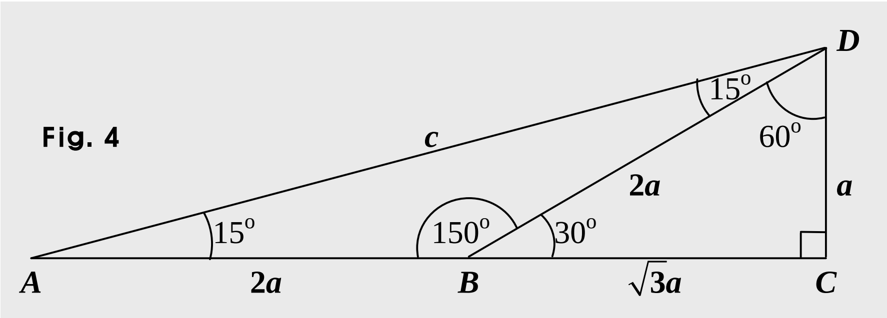
It is an isosceles triangle, where two of the three angles are 15° each.
So we may want to put labels a bit differently, indicating angles and sides. And we can put them the way below.

Fig. 3



How do we get though, $\boldsymbol{BC} = \sqrt{3}\boldsymbol{a}$, and $\boldsymbol{BD} = 2\boldsymbol{a}$?

We know the triangle ***BCD*** is half the regular triangle where each side is $2\boldsymbol{a}$.



So $\boldsymbol{BD = 2a}$. And we can get the length of $\boldsymbol{BC}$ using the distance formula, often called the Pythagorean theorem, too.

Assuming thus, $\boldsymbol{x}$ is the length of $\boldsymbol{BC}$, and using the formula, we get

$$(BD)^2 = (BC)^2 + (CD)^2 \Rightarrow 4a^2 = x^2 + a^2 \Rightarrow x^2 = 4a^2 - a^2 = 3a^2 \Rightarrow x = \pm\sqrt{3}a.$$

And we know $\boldsymbol{x}$ is a length, which is $\geq \mathbf{0}$. So we get $\boldsymbol{x = \sqrt{3}a}$. And in fact, in a half a regular triangle, the ratio between the three sides is $\mathbf{1 : 2 : \sqrt{3}}$. What then is the next?

We can get the length of the side $\boldsymbol{AC}$, which is the adjacent if the angle $\boldsymbol{A}$ is the governing angle. And the length is $\boldsymbol{2a + \sqrt{3}a}$, which is the adjacent in the triangle $\boldsymbol{ACD}$.

Now, we know the cosine is the adjacent over the hypotenuse.
So what do we want to find?

It is the hypotenuse, which is $\boldsymbol{c}$ in the right triangle $\boldsymbol{ACD}$ above.
How then can we get the value of $\boldsymbol{c}$?

We can get it using the distance formula. too. So using the formula again, we can get

$$(AD)^2 = (AC)^2 + (CD)^2 \Rightarrow c^2 = (2a + \sqrt{3}a)^2 + a^2 = 4a^2 + 4\sqrt{3}\,a^2 + 3a^2 + a^2 = (8 + 4\sqrt{3}\,)a^2.$$

So we get $\boldsymbol{c^2 = (8+4\sqrt{3}\,)a^2}$. Thus, we get $\boldsymbol{c = a\sqrt{8+4\sqrt{3}}}$, since $\boldsymbol{c}$ is a length.

Doing some algebra though, we can put $\boldsymbol{c}$ the way below, too.

$$8+4\sqrt{3}=8+2\cdot 2\sqrt{3}=8+2\sqrt{4}\sqrt{3}=8+2\sqrt{12}=8+2\sqrt{2\cdot 6}=8+2\sqrt{2}\sqrt{6}$$

$$=2+6+2\sqrt{2}\sqrt{6}=(\sqrt{2}+\sqrt{6})^2 \Rightarrow 8+4\sqrt{3}=(\sqrt{2}+\sqrt{6})^2 \Rightarrow c=(\sqrt{2}+\sqrt{6})a.$$

Now, we know that the cosine is the adjacent over the hypotenuse.

And the adjacent is $\mathbf{2a+\sqrt{3}a}$, which is $\mathbf{(2+\sqrt{3})a}$, and the hypotenuse is $\mathbf{(\sqrt{2}+\sqrt{6})a}$.

So we can put the cosine of 15^o, that is, **cos** 15^o the way as follows.

$$\cos 15^o = \frac{(2+\sqrt{3})a}{(\sqrt{2}+\sqrt{6})a}=\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{6}}=\frac{(2+\sqrt{3})}{\sqrt{2}(1+\sqrt{3})}=\frac{(2+\sqrt{3})(\sqrt{3}-1)}{\sqrt{2}(1+\sqrt{3})(\sqrt{3}-1)}$$

$$=\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}+\sqrt{6}}{4}\Rightarrow \cos 15^\circ=\frac{\sqrt{2}+\sqrt{6}}{4}\approx 0.96592582628906828.$$

Isn't there any other way though, to find the length of the line segment ***AD***?

We can get it using a formula called the cosine rule, too, which will be covered in the section called **The Cosine Rule** in the book **Trigonometry**.

And assuming θ is the angle ***ABD*** in the triangle ***ABD*** above, and using the cosine rule, we get $(AD)^2 = (AB)^2 + (BD)^2 - 2(AB)(BD)\cos\theta$.

So we get $c^2 = 4a^2 + 4a^2 - 2\cdot 2a\cdot 2a\cdot \cos 150^o = 8a^2 + 2\cdot 4a^2\cdot \cos 30^o = (8+4\sqrt{3})a^2$.