Examples in The Sine Rule

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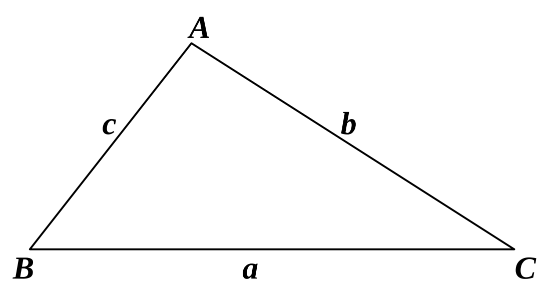
Examples in The Sine Rule

Doing all these examples, refer to the triangle ***ABC*** below.

Fig. 0



0. Show that
sin *A* + sin *B* > sin *C*, sin *B* + sin *C* > sin *A*, and sin *C* + sin *A* > sin *B*.

1. Assuming $\boldsymbol{2b = a + c}$, show the relation among the angles $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.

2. Assuming $\boldsymbol{A : B : C = 3 : 4 : 5}$, find $\boldsymbol{a : b : c}$.

3. The equation below is for $\boldsymbol{x}$, is quadratic, and has two identical solutions, that is, a double root. What kind of triangle then is the triangle ***ABC***?

$$\boldsymbol{(\sin C + \cos A)x^2 + (2\cos B)x - (\sin C - \cos A) = 0}.$$

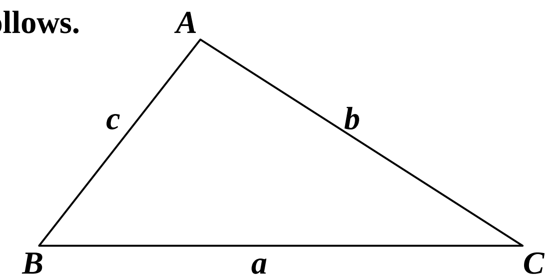
4. Assuming $\boldsymbol{a = 100}$, $\boldsymbol{B = \pi/3}$, and $\boldsymbol{C = 5\pi/12}$, find $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

5. Assuming $\boldsymbol{b = 15}$, $\boldsymbol{c = 15\sqrt{3}}$, and $\boldsymbol{B = \pi/6}$, find $\boldsymbol{A}$, $\boldsymbol{a}$, and $\boldsymbol{C}$.

Suggestions or Solutions
To the Problem in the Example 0

Show that sin *A* + sin *B* > sin *C*, sin *B* + sin *C* > sin *A*, and sin *C* + sin *A* > sin *B* for the triangle *ABC* as follows.

Fig. 0.0



We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, $a + b > c$, $b + c > a$, and $c + a > b$.

So first, we get $a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$.

Thus, next, we get

$a + b > c \Rightarrow 2R \sin A + 2R \sin B > 2R \sin C \Rightarrow \sin A + \sin B > \sin C$.

$b + c > a \Rightarrow 2R \sin B + 2R \sin C > 2R \sin A \Rightarrow \sin B + \sin C > \sin A$.

$c + a > b \Rightarrow 2R \sin C + 2R \sin A > 2R \sin B \Rightarrow \sin C + \sin A > \sin B$.

If not quite sure of the idea behind the processes above, follow the steps below.

Each of the relations given looks like a property in a triangle. What property?

It is called Triangle Inequality, and says that the sum of two sides is grater than the other side, in a triangle, of course. So for instance, we get $a + b > c$, $b + c > a$, and $c + a > b$.

How then can use the fact, the inequality above?

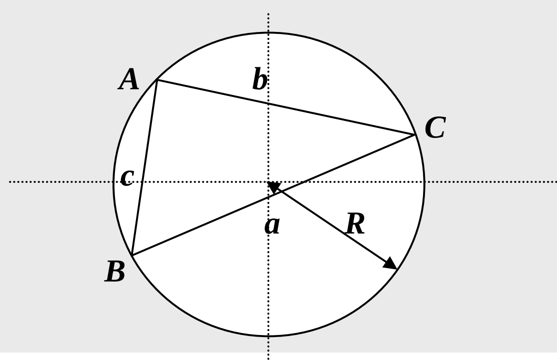
Looking at the relations given, we can notice that they are made of the sines. So we may want to try the sine rule. What then is the sine rule?

Referring to the triangle above, we can put it this way: $\dfrac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \dfrac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \dfrac{\boldsymbol{c}}{\sin \boldsymbol{C}} = 2\boldsymbol{R}$.

What then is ***R***?

It is the radius of a circle called a circumcircle, which passes through all the vertices of the triangle ***ABC***, which is thus, circumscribed by the circle of radius ***R***.
So putting the triangle ***ABC*** in the circle, we can put it the way below.

Fig. 0.1



How then can we use the sine rule?

We can extract the three sides from the sine rule, and set up the inequality.

And the sine rule is $\dfrac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \dfrac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \dfrac{\boldsymbol{c}}{\sin \boldsymbol{C}} = 2\boldsymbol{R}$.

So we get $\boldsymbol{a} = 2\boldsymbol{R} \sin \boldsymbol{A}$, $\boldsymbol{b} = 2\boldsymbol{R} \sin \boldsymbol{B}$, and $\boldsymbol{c} = 2\boldsymbol{R} \sin \boldsymbol{C}$.

And setting up the inequality, we get $\boldsymbol{a} + \boldsymbol{b} > \boldsymbol{c} \Rightarrow 2\boldsymbol{R} \sin \boldsymbol{A} + 2\boldsymbol{R} \sin \boldsymbol{B} > 2\boldsymbol{R} \sin \boldsymbol{C}$.

We know $2\boldsymbol{R}$ is positive. So dividing by $2\boldsymbol{R}$ both sides, we get $\sin \boldsymbol{A} + \sin \boldsymbol{B} > \sin \boldsymbol{C}$.
And the same is true, also, for the other two relations.
So setting up again, the inequality, we get $\boldsymbol{b} + \boldsymbol{c} > \boldsymbol{a} \Rightarrow 2\boldsymbol{R} \sin \boldsymbol{B} + 2\boldsymbol{R} \sin \boldsymbol{C} > 2\boldsymbol{R} \sin \boldsymbol{A}$.

And dividing by $2R$ both sides, we get $\sin B + \sin C > \sin A$.

Setting up again, the inequality, we get $c + a > b \Rightarrow 2R \sin C + 2R \sin A > 2R \sin B$.

And dividing again, by $2R$ both sides, we get $\sin C + \sin A > \sin B$.

In short:

We have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, $a + b > c$, $b + c > a$, and $c + a > b$.

So first, we get $a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$.
Thus, next, we get

$a + b > c \Rightarrow 2R \sin A + 2R \sin B > 2R \sin C \Rightarrow \sin A + \sin B > \sin C$.

$b + c > a \Rightarrow 2R \sin B + 2R \sin C > 2R \sin A \Rightarrow \sin B + \sin C > \sin A$.

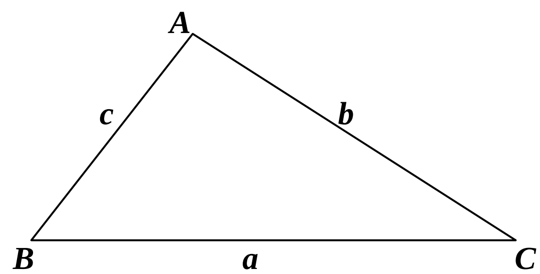
$c + a > b \Rightarrow 2R \sin C + 2R \sin A > 2R \sin B \Rightarrow \sin C + \sin A > \sin B$.

Suggestions or Solutions

To the Problem in the Example 1

Assuming 2*b* = *a* + *c*, show the relation among the angles *A*, *B* and *C* for the triangle as follows.

Fig. 1.0



What do we mean by showing the relation among the angles A, B and C?

We can show it producing a math expression in terms of A, B and C.
We know that A, B and C are angles. How then can we relate angles?

Using an angle, we can express a ratio. What ratio, though?

It is a trig-ratio as the sine, the cosine, and the tangent.
So we can try using such a ratio setting up the relation among the angles A, B and C.
We cannot however, just set up a relation. The relation has to be on a specific basis.
What can be the basis then?

We are given a relation between the sides in the triangle that has the angles. And the relation is the expression where $2b = a + c$. So we want to use it. How though?

We have the sine rule, and we can extract the three sides from the sine rule, and set up the relation. What then is the sine rule?

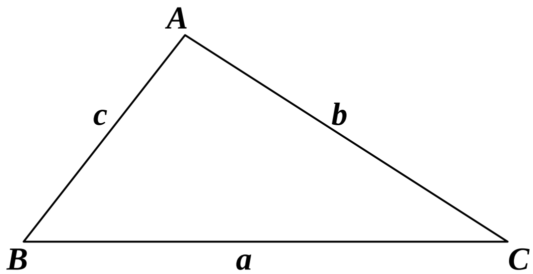
The sine rule is $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

So we get $a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$.
And setting up the relation, we get $2b = a + c \Rightarrow 2 \cdot 2R \sin B = 2R \sin A + 2R \sin C$.
We know $2R$ is positive. So dividing by $2R$ both sides, we get $2\sin B = \sin A + \sin C$.

Suggestions or Solutions
To the Problem in the Example 2

Find $a : b : c$ assuming $A : B : C = 3 : 4 : 5$ for the triangle ABC below:

Fig. 2.0



We want to find the ratio between the sides in a triangle, based on the ratio between the angles in the triangle.
So it seems we want to make use of the ratio between the angles. How?

We can put the ratio in terms of the angles. How?

> We have the sine rule where $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

So we can get $a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$.
Thus, we can for now, put the ratio the way below.

$a : b : c = 2R \sin A : 2R \sin B : 2R \sin C = \sin A : \sin B : \sin C$.

So we now have $a : b : c = \sin A : \sin B : \sin C$. What then is the next?

We have not use the fact given, and the fact is the ratio between the angles. So we want to make use of the ratio, which is $A : B : C = 3 : 4 : 5$. How then can we use it?

We have another property in a triangle. And the property is the fact that the sum of the three angles in a triangle is 180^{o}, which is π.

So using the fact above, we can actually get the values of the angles, A, B, and C.

And finding them, we get $\boldsymbol{A} = \pi\frac{3}{3+4+5} = \pi\frac{3}{12} = \pi/4$, $\boldsymbol{B} = \pi\frac{4}{12} = \pi/3$, and $\boldsymbol{C} = 5\pi/12$.

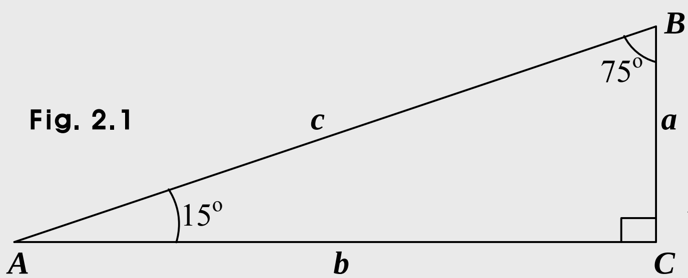
So we get $\boldsymbol{a : b : c} = \mathbf{sin}\ \pi/4 : \mathbf{sin}\ \pi/3 : \mathbf{sin}\ 5\pi/12$.

And we know $\mathbf{sin}\ \pi/4 = \frac{\sqrt{2}}{2}$, and $\mathbf{sin}\ \pi/3 = \frac{\sqrt{3}}{2}$. What about **sin 5π/12** though?

We know 5π/12 is the governing angle.

So using a right triangle where an angle is 5π/12, which is 75°, we should be able to get the value of the trig ratio, **sin 75°**. Thus, forming such a triangle, we can get this:

Fig. 2.1



We know that the sine is the opposite over the hypotenuse.

So we can set $\mathbf{sin}\ 75^{\circ} = \boldsymbol{b/c}$.

And in fact, we found the ratio ***b/c*** in the example 0 in the section, **Triangles and Trigonometry** in the book **Trigonometry**.

And the ratio is $\frac{\sqrt{2}+\sqrt{6}}{4}$. So we get $\mathbf{sin}\ 75^{\circ} = \frac{\sqrt{2}+\sqrt{6}}{4}$.

Thus, we now can set $\boldsymbol{a : b : c} = \frac{\sqrt{2}}{2} : \frac{\sqrt{3}}{2} : \frac{\sqrt{2}+\sqrt{6}}{4}$, which is $2\sqrt{2} : 2\sqrt{3} : \sqrt{2}+\sqrt{6}$.

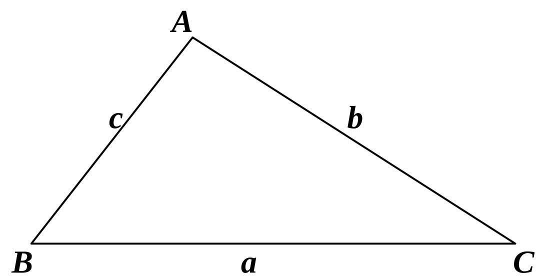
So we get $\boldsymbol{a : b : c} = 2\sqrt{2} : 2\sqrt{3} : \sqrt{2}+\sqrt{6}$.

Suggestions or Solutions
To the Problem in the Example 3

The equation below is for x, is quadratic, and has two identical solutions, that is, a double root. What kind of triangle then is the triangle ABC?

$$(\sin C + \cos A)x^2 + (2\cos B)x - (\sin C - \cos A) = 0.$$

Fig. 3.0



Assuming D is the discriminant, we get

$D = (2\cos B)^2 - 4(\sin C + \cos A)\{-(\sin C - \cos A)\} = 0$

$\Rightarrow 4\cos^2 B + 4(\sin C + \cos A)(\sin C - \cos A) = 0$.

$\Rightarrow \cos^2 B + (\sin C + \cos A)(\sin C - \cos A) = 0 \Rightarrow \cos^2 B + \sin^2 C - \cos^2 A = 0$.

And we have a trig identity where $\sin^2 \theta + \cos^2 \theta = 1$. So we get

$\sin^2 A + \cos^2 A = 1 \Rightarrow \cos^2 A = 1 - \sin^2 A$, and

$\sin^2 B + \cos^2 B = 1 \Rightarrow \cos^2 B = 1 - \sin^2 B$. Thus, we get

$\cos^2 B + \sin^2 C - \cos^2 A = 1 - \sin^2 B + \sin^2 C - (1 - \sin^2 A)$

$\Rightarrow \sin^2 A + \sin^2 C = \sin^2 B$.

Also, we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

So we get $\sin A = \dfrac{a}{2R}$, $\sin B = \dfrac{b}{2R}$, and $\sin C = \dfrac{c}{2R}$.

Thus, we get $\sin^2 A + \sin^2 C = \sin^2 B \Rightarrow \left(\dfrac{a}{2R}\right)^2 + \left(\dfrac{c}{2R}\right)^2 = \left(\dfrac{b}{2R}\right)^2 \Rightarrow a^2 + c^2 = b^2$.

So the triangle ABC is a right triangle where b is the hypotenuse.

If not quite sure of the idea behind the processes above, follow the steps below.

If a quadratic equation $\boldsymbol{ux^2 + vx + w = 0}$ has a double root, we get $\boldsymbol{D = 0}$, where $\boldsymbol{D}$ is called the discriminant. And $\boldsymbol{D}$ is in this case, $\boldsymbol{v^2 - 4uw}$.
So since the equation given has a double root, applying the discriminant $\boldsymbol{D}$ to the equation, we get $\boldsymbol{D = (2\cos B)^2 - 4(\sin C + \cos A)\{-(\sin C - \cos A)\} = 0}$.

That is, we get $\boldsymbol{4\cos^2 B + 4(\sin C + \cos A)(\sin C - \cos A) = 0}$.
And dividing by 4 both sides, we get $\boldsymbol{\cos^2 B + (\sin C + \cos A)(\sin C - \cos A) = 0}$.
And simplifying it, we get $\boldsymbol{\cos^2 B + \sin^2 C - \cos^2 A = 0}$.
How then do we know the kind the triangle $\boldsymbol{ABC}$ belongs to?

Finding the relation among the sides, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, or knowing about the angles, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, we can see the kind. In this case, we can find the relation among the sides easier.
How though?

We can use two facts, one is a trig identity, and the other is the sine rule.
What trig identity?

It is a trig identity where $\boldsymbol{\sin^2 \theta + \cos^2 \theta = 1}$. So we get
$\boldsymbol{\sin^2 A + \cos^2 A = 1 \Rightarrow \cos^2 A = 1 - \sin^2 A}$, and

$\boldsymbol{\sin^2 B + \cos^2 B = 1 \Rightarrow \cos^2 B = 1 - \sin^2 B}$. Thus, we get

$$\boldsymbol{\cos^2 B + \sin^2 C - \cos^2 A = 1 - \sin^2 B + \sin^2 C - (1 - \sin^2 A) \Rightarrow \sin^2 A + \sin^2 C = \sin^2 B}.$$

And moving on to the sine rule, we have $\boldsymbol{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R}$.

So we get $\boldsymbol{\sin A = \dfrac{a}{2R}}$, $\boldsymbol{\sin B = \dfrac{b}{2R}}$, and $\boldsymbol{\sin C = \dfrac{c}{2R}}$.

Thus, we get $\boldsymbol{\sin^2 A + \sin^2 C = \sin^2 B \Rightarrow \left(\dfrac{a}{2R}\right)^2 + \left(\dfrac{c}{2R}\right)^2 = \left(\dfrac{b}{2R}\right)^2}$.

And multiplying by $\boldsymbol{4R^2}$ both sides, we get $\boldsymbol{a^2 + c^2 = b^2}$, which indicates a right triangle where $\boldsymbol{b}$ is the hypotenuse.

So the triangle ***ABC*** is a right triangle where ***b*** is the hypotenuse, and ***B*** is a right angle, since the hypotenuse faces the right angle.

In short:

Assuming ***D*** is the discriminant, we get

$\boldsymbol{D = (2\cos B)^2 - 4(\sin C + \cos A)\{-(\sin C - \cos A)\} = 0}$

$\boldsymbol{\Rightarrow 4\cos^2 B + 4(\sin C + \cos A)(\sin C - \cos A) = 0}$.

$\boldsymbol{\Rightarrow \cos^2 B + (\sin C + \cos A)(\sin C - \cos A) = 0 \Rightarrow \cos^2 B + \sin^2 C - \cos^2 A = 0}$.

And we have a trig identity where $\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$. So we get

$\boldsymbol{\sin^2 A + \cos^2 A = 1 \Rightarrow \cos^2 A = 1 - \sin^2 A}$, and

$\boldsymbol{\sin^2 B + \cos^2 B = 1 \Rightarrow \cos^2 B = 1 - \sin^2 B}$. Thus, we get

$\boldsymbol{\cos^2 B + \sin^2 C - \cos^2 A = 1 - \sin^2 B + \sin^2 C - (1 - \sin^2 A) \Rightarrow \sin^2 A + \sin^2 C = \sin^2 B}$.

Also, we have $\boldsymbol{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R}$.

So we get $\boldsymbol{\sin A = \dfrac{a}{2R}}$, $\boldsymbol{\sin B = \dfrac{b}{2R}}$, and $\boldsymbol{\sin C = \dfrac{c}{2R}}$.

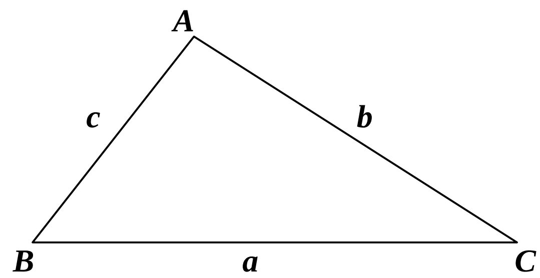
Thus, we get $\boldsymbol{\sin^2 A + \sin^2 C = \sin^2 B \Rightarrow \left(\dfrac{a}{2R}\right)^2 + \left(\dfrac{c}{2R}\right)^2 = \left(\dfrac{b}{2R}\right)^2 \Rightarrow a^2 + c^2 = b^2}$.

So the triangle ***ABC*** is a right triangle where ***b*** is the hypotenuse.

Suggestions or Solutions
To the Problem in the Example 4

Find A, b, and c assuming $a = 100$, $B = \pi/3$, and $C = 5\pi/12$ for the triangle below.

Fig. 4.0



To begin with, we know two angles in a triangle.
So we can get the other angle, since the sum of the three angles in a triangle is π.
And the two are $B = \pi/3$, and $C = 5\pi/12$.
Thus, we get $A = \pi - B - C = \pi - \pi/3 - 5\pi/12 = \pi - 4\pi/12 - 5\pi/12 = \pi - 9\pi/12 = 3\pi/12$.

So we get $A = \pi/4$. What then is the next?

Next, we have the sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

So we get $\dfrac{100}{\sin \pi/4} = \dfrac{b}{\sin \pi/3} = \dfrac{c}{\sin 5\pi/12} = 2R$. What then?

We know $\sin \pi/4 = \frac{\sqrt{2}}{2}$, $\sin \pi/3 = \frac{\sqrt{3}}{2}$, and $\sin 5\pi/12 = \frac{\sqrt{2}+\sqrt{6}}{4}$, from the example 2.

So first, we can get $\dfrac{100}{\sin\dfrac{\pi}{4}} = \dfrac{b}{\sin\dfrac{\pi}{3}}$

$$\Rightarrow b = \frac{100\sin\dfrac{\pi}{3}}{\sin\dfrac{\pi}{4}} = \frac{100\,\frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \frac{100\sqrt{3}}{\sqrt{2}} = \frac{100\sqrt{6}}{2} = 50\sqrt{6} \quad \Rightarrow b = 50\sqrt{6}.$$

And next, we can get $\dfrac{100}{\sin\frac{\pi}{4}} = \dfrac{c}{\sin\frac{5\pi}{12}}$

$$\Rightarrow c = \frac{100\sin\frac{5\pi}{12}}{\sin\frac{\pi}{4}} = \frac{100\,\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{\sqrt{2}}{2}} = \frac{50(\sqrt{2}+\sqrt{6})}{\sqrt{2}} = \frac{50\sqrt{2}(\sqrt{2}+\sqrt{6})}{2}$$

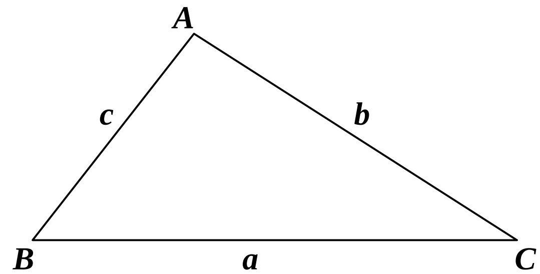
$$= 25(2+2\sqrt{3}) = 50(1+\sqrt{3}) \;\; \Rightarrow c = 50(1+\sqrt{3}).$$

Suggestions or Solutions

To the Problem in the Example 5

Find A, a, and C assuming $b = 15$, $c = 15\sqrt{3}$, and $B = \pi/6$ for the triangle below'

Fig. 5.0



To begin with, we have the sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

So we get $\dfrac{a}{\sin A} = \dfrac{15}{\sin\frac{\pi}{6}} = \dfrac{15\sqrt{3}}{\sin C} = 2R$. And we know $\sin \pi/6 = 1/2$. So?

So first, we can get $\dfrac{15}{\sin\frac{\pi}{6}} = \dfrac{15\sqrt{3}}{\sin C} \Rightarrow 15\sin C = 15\sqrt{3}\sin\dfrac{\pi}{6} = \frac{15\sqrt{3}}{2} \Rightarrow \sin C = \frac{\sqrt{3}}{2}$.

Thus, we get $C = \pi/3$ or $\pi - \pi/3 = 2\pi/3$, since we have $\sin(\pi - \theta) = \sin\theta$ for $0 \le \theta \le \pi/2$.
And we know the sum of the three angles in a triangle is π. So?

So assuming first, $C = \pi/3$, we get $A + B + C = A + \pi/6 + \pi/3 = \pi \Rightarrow A = \pi/2$.

So next, we can get $\dfrac{a}{\sin\frac{\pi}{2}} = \dfrac{15}{\sin\frac{\pi}{6}} = \dfrac{15}{\frac{1}{2}} = 30 \Rightarrow a = 30\sin\pi/2 = 30$, for $\sin\pi/2 = 1$.

And assuming next, $C = 2\pi/3$, we get $A + B + C = A + \pi/6 + 2\pi/3 = \pi \Rightarrow A = \pi/6$.

And next, we can get $\dfrac{a}{\sin\frac{\pi}{6}} = \dfrac{15}{\sin\frac{\pi}{6}} \Rightarrow a = 15$.

Thus, we get $A = \pi/2$, $a = 30$, and $C = \pi/3$. $A = \pi/6$, $a = 15$, and $C = 2\pi/3$.