Examples 1 in The Cosine Rule

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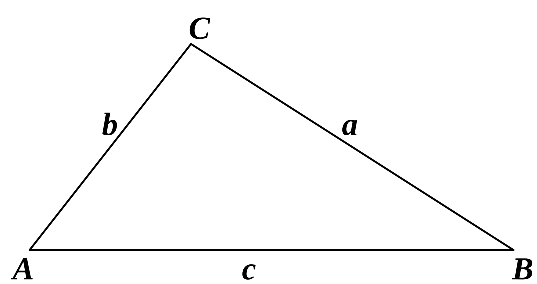
Doing all these examples, refer to the triangle $\boldsymbol{ABC}$ below.

Fig. 0



0. Show that $\mathbf{0} < \boldsymbol{A} < \boldsymbol{\pi/2} \Rightarrow \boldsymbol{a}^2 < \boldsymbol{b}^2 + \boldsymbol{c}^2$.

1. Show that $\boldsymbol{\pi/2} < \boldsymbol{A} < \boldsymbol{\pi} \Rightarrow \boldsymbol{a}^2 > \boldsymbol{b}^2 + \boldsymbol{c}^2$

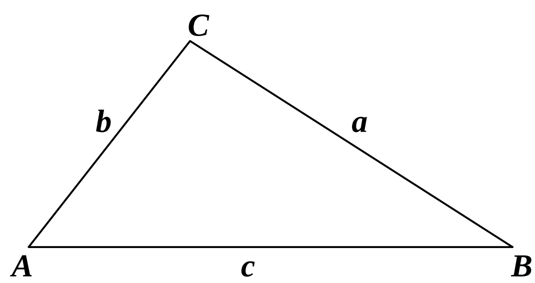
2. Assuming $\boldsymbol{A} = \boldsymbol{\pi/3}$, $\boldsymbol{b} = \mathbf{40}$, and $\boldsymbol{c} = \mathbf{20}(\sqrt{\mathbf{3}} + \mathbf{1})$, find $\boldsymbol{a}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

3. Assuming $\boldsymbol{a} = \sqrt{\mathbf{6}}$, $\boldsymbol{b} = \mathbf{2}\sqrt{\mathbf{3}}$, and $\boldsymbol{c} = \mathbf{3} + \sqrt{\mathbf{3}}$, find $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$.

Suggestions or Solutions
To the Problem in the Example 0

Show that $0 < A < \pi/2 \Rightarrow a^2 < b^2 + c^2$ for the triangle *ABC* below.

Fig. 0.0



We have $a^2 = b^2 + c^2 - 2bc \cos A$.

Also, we have $bc > 0$, and $0 < A < \pi/2 \Rightarrow \cos A > 0$.

So we get $b^2 + c^2 - a^2 = 2bc \cos A > 0 \Rightarrow b^2 + c^2 - a^2 > 0 \Rightarrow b^2 + c^2 > a^2$.

Thus, we get $0 < A < \pi/2 \Rightarrow a^2 < b^2 + c^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

The relation given looks like the triangle inequality, and the inequality says
$a + b > c$, $b + c > a$, or $c + b > a$.

The inequality does not though, explain the relation given.

We have however, a trig identity that shows a relation between the squares of the sides of a triangle. What then is the trig identity?

It is the cosine rule, where

$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$, $\boldsymbol{b^2 = c^2 + a^2 - 2ca \cos B}$, or $\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C}$.

So using the cosine rule, along with $\boldsymbol{0 < A < \pi/2}$, we can begin with an equation below.
$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$.

And next, we can put it this way: $\boldsymbol{b^2 + c^2 - a^2 = 2bc \cos A}$.

Also, we know that $\boldsymbol{0 < A < \pi/2 \Rightarrow \cos A > 0}$, and that $\boldsymbol{bc > 0}$ since $\boldsymbol{b > 0}$ and $\boldsymbol{c > 0}$.

So we get $\boldsymbol{b^2 + c^2 - a^2 = 2bc \cos A > 0}$.
That is to say that we can get $\boldsymbol{b^2 + c^2 - a^2 > 0}$. Thus, we get $\boldsymbol{b^2 + c^2 > a^2}$.

So we get $\boldsymbol{0 < A < \pi/2 \Rightarrow a^2 < b^2 + c^2}$.

In short:

We have $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$.

Also, we have $\boldsymbol{bc > 0}$, and $\boldsymbol{0 < A < \pi/2 \Rightarrow \cos A > 0}$.

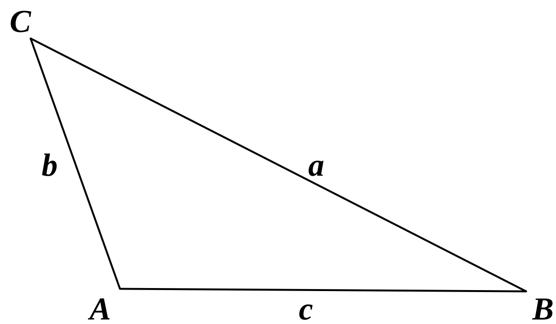
So we get $\boldsymbol{b^2 + c^2 - a^2 = 2bc \cos A > 0 \Rightarrow b^2 + c^2 - a^2 > 0 \Rightarrow b^2 + c^2 > a^2}$.

Thus, we get $\boldsymbol{0 < A < \pi/2 \Rightarrow a^2 < b^2 + c^2}$.

Suggestions or Solutions
To the Problem in the Example 1

Show that $\pi/2 < A < \pi \Rightarrow a^2 > b^2 + c^2$ for the triangle *ABC* below.

Fig. 1.0



We have $a^2 = b^2 + c^2 - 2bc \cos A$.

Also, we have $bc > 0$, and $\pi/2 < A < \pi \Rightarrow \cos A < 0$.

So we get $b^2 + c^2 - a^2 = 2bc \cos A < 0 \Rightarrow b^2 + c^2 - a^2 < 0 \Rightarrow b^2 + c^2 < a^2$.

Thus, we get $\pi/2 < A < \pi \Rightarrow a^2 > b^2 + c^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

This example is no other than the example 0 covered earlier.

So we can use again, the cosine rule, where

$a^2 = b^2 + c^2 - 2bc \cos A$, $b^2 = c^2 + a^2 - 2ca \cos B$, or $c^2 = a^2 + b^2 - 2ab \cos C$.

Using thus, the cosine rule, along with $0 < A < \pi/2$, we can begin with an equation as follows: $a^2 = b^2 + c^2 - 2bc \cos A$.

And next, we can put it this way again: $\boldsymbol{b^2 + c^2 - a^2 = 2bc \cos A}$. What then?

We know that $\pi/\mathbf{2} < \boldsymbol{A} < \pi \Rightarrow \mathbf{cos}\ \boldsymbol{A} < \mathbf{0}$, and that $\boldsymbol{bc} > \mathbf{0}$.

So we get $\boldsymbol{b^2 + c^2 - a^2 = 2bc \cos A < 0}$.

That is, we can get $\boldsymbol{b^2 + c^2 - a^2 < 0}$. Thus, we get $\boldsymbol{b^2 + c^2 < a^2}$.

So we get $\pi/\mathbf{2} < \boldsymbol{A} < \pi \Rightarrow \boldsymbol{a^2 > b^2 + c^2}$.

In short:

We have $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$.

Also, we have $\boldsymbol{bc} > \mathbf{0}$, and $\pi/\mathbf{2} < \boldsymbol{A} < \pi \Rightarrow \mathbf{cos}\ \boldsymbol{A} < \mathbf{0}$.

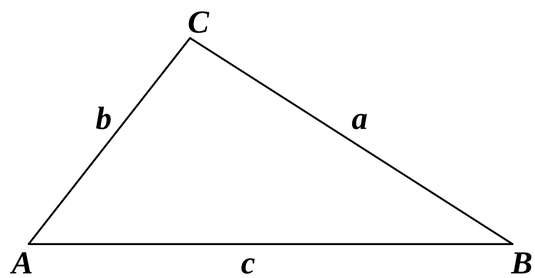
So we get $\boldsymbol{b^2 + c^2 - a^2 = 2bc \cos A < 0 \Rightarrow b^2 + c^2 - a^2 < 0 \Rightarrow b^2 + c^2 < a^2}$.

Thus, we get $\pi/\mathbf{2} < \boldsymbol{A} < \pi \Rightarrow \boldsymbol{a^2 > b^2 + c^2}$.

Suggestions or Solutions
To the Problem in the Example 2

Find a, B, and C assuming $A = \pi/3$, $b = 40$, and $c = 20(\sqrt{3}+1)$ for the triangle ABC as follows.

Fig. 2.0



$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow a^2 = 40^2 + 20^2(\sqrt{3}+1)^2 - 2\cdot 40\cdot 20(\sqrt{3}+1)\cos \pi/3$

$= 1600 + 400(3 + 2\sqrt{3} + 1) - 1600(\sqrt{3}+1)\cdot(1/2) = 2400 \Rightarrow a^2 = 2400 \Rightarrow a = \pm 20\sqrt{6}$.

So we get $a = 20\sqrt{6}$ since $a > 0$.

Next, using the sine rule, we get $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{20\sqrt{6}}{\frac{\sqrt{3}}{2}} = 40\sqrt{2} = \dfrac{40}{\sin B}$.

So we get $\sin B = \dfrac{1}{\sqrt{2}}$. Thus, we get $B = \pi/4$ or $\pi - \pi/4 = 3\pi/4$.

Assuming however, $B = \pi - \pi/4 = 3\pi/4$, we get $A + B = \pi/3 + 3\pi/4 = 13\pi/12 > \pi$, which is not possible, because $A + B + C = \pi$, and $C > 0$.

So we get $B \neq 3\pi/4$, and $B = \pi/4$.
And we have $A + B + C = \pi$.
Thus, we get $C = \pi - A - B = \pi - \pi/3 - \pi/4 = 5\pi/12$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, knowing two angles in a triangle, we can get the other angle, since the sum of all the three angles is π.

In this case however, we know one angle only, which is $\boldsymbol{A}$.
So how can we find the other two angles, $\boldsymbol{B}$ and $\boldsymbol{C}$?

We know the angle $\boldsymbol{A}$, and two sides, $\boldsymbol{b}$ and $\boldsymbol{c}$. And we have the sine rule below.

$\dfrac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \dfrac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \dfrac{\boldsymbol{c}}{\sin \boldsymbol{C}} = 2\boldsymbol{R}$. So?

So finding the side $\boldsymbol{a}$, we can get the angles $\boldsymbol{B}$ and $\boldsymbol{C}$ both.

How then can we get the side $\boldsymbol{a}$?

We know that the angle $\boldsymbol{A}$ is between the two sides $\boldsymbol{b}$ and $\boldsymbol{c}$.
So what can we use to find the side $\boldsymbol{a}$?

We have the cosine rule, where $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$.
So using the cosine rule, we can get the side $\boldsymbol{a}$. Using thus, the rule, we get

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow a^2 = 40^2 + 20^2(\sqrt{3}+1)^2 - 2\cdot 40\cdot 20(\sqrt{3}+1)\cos \pi/3}$$

$$\boldsymbol{= 1600 + 400(3 + 2\sqrt{3} + 1) - 1600(\sqrt{3}+1)\cdot(1/2) = 1600 + 1600 - 800 = 2400}$$

$$\boldsymbol{\Rightarrow a^2 = 2400 \Rightarrow a = \pm 20\sqrt{6}}.$$

So we get $\boldsymbol{a = 20\sqrt{6}}$, since $\boldsymbol{a > 0}$. What then can we do with the value of $\boldsymbol{a}$?

We can get the value of the angle $\boldsymbol{B}$ using the sine rule.

Thus next, using the sine rule, we get $\dfrac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \dfrac{\boldsymbol{b}}{\sin \boldsymbol{B}} \Rightarrow \dfrac{\boldsymbol{20\sqrt{6}}}{\frac{\sqrt{3}}{2}} = \dfrac{\boldsymbol{40}}{\sin \boldsymbol{B}}$.

And we have $\dfrac{\boldsymbol{20\sqrt{6}}}{\frac{\sqrt{3}}{2}} = \dfrac{\boldsymbol{2\cdot 20\sqrt{2}\sqrt{3}}}{\sqrt{3}} = \boldsymbol{40\sqrt{2}}$.

So we get $\frac{\mathbf{40}}{\mathbf{\sin}\,\boldsymbol{B}} = \mathbf{40}\sqrt{\mathbf{2}} \Rightarrow \mathbf{\sin}\,\boldsymbol{B} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$.

Thus, we get $\boldsymbol{B} = \pi/4$ or $\pi - \pi/4$, since we have $\sin(\pi - \theta) = \sin\theta$ for $0 \leq \theta \leq \pi/2$.

Assuming however, $\boldsymbol{B} = \pi - \pi/4 = 3\pi/4$, we get $\boldsymbol{A} + \boldsymbol{B} = \pi/3 + 3\pi/4 = 13\pi/12 > \pi$, which is not possible, since $\boldsymbol{A}$ and $\boldsymbol{B}$ are two angles in the triangle $\boldsymbol{ABC}$, and the sum of the three angles in a triangle is π, that is, $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = \pi$.

So we get $\boldsymbol{B} \neq 3\pi/4$, and $\boldsymbol{B} = \pi/4$.

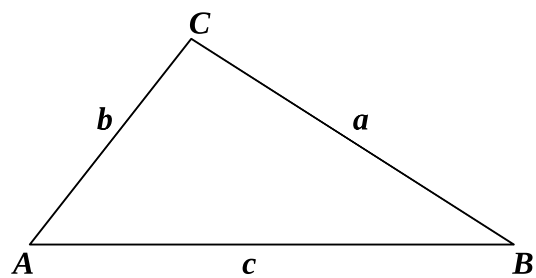
Thus, since $\boldsymbol{B} = \pi/4$, and $\boldsymbol{A} = \pi/3$, we get $\boldsymbol{C} = \pi - \boldsymbol{A} - \boldsymbol{B} = \pi - \pi/3 - \pi/4 = 5\pi/12$.

Suggestions or Solutions

To the Problem in the Example 3

Find A, B, and C assuming $a = \sqrt{6}, b = 2\sqrt{3}$, and $c = 3 + \sqrt{3}$ for the triangle below.

Fig. 3.0



We have all the three sides in a triangle, but have none of the three angles.
So what can we use to find all the three angles?

We have a formula where an angle in triangle can be expressed in terms of the sides.
What then is the formula?

The formula is in fact, called a rule, and is known as the cosine rule. And the rule says

$a^2 = b^2 + c^2 - 2bc \cos A$, $b^2 = c^2 + a^2 - 2ca \cos B$, and $c^2 = a^2 + b^2 - 2ab \cos C$.

Knowing thus, the three sides, we can get all the three angles.
In other words, we can get

$$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$b^2 = c^2 + a^2 - 2ca \cos B \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

$$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Thus, beginning with the angle A, we get

$$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 6 = 12 + (3 + \sqrt{3})^2 - 2 \cdot 2\sqrt{3}(3 + \sqrt{3}) \cos A$$

$$= 12 + 9 + 6\sqrt{3} + 3 - (12\sqrt{3} + 12)\cos A = 24 + 6\sqrt{3} - 12(\sqrt{3}+1)\cos A$$

$$\Rightarrow \cos A = \frac{18+6\sqrt{3}}{12(\sqrt{3}+1)} = \frac{6(3+\sqrt{3})}{12(\sqrt{3}+1)} = \frac{\sqrt{3}(\sqrt{3}+1)}{2(\sqrt{3}+1)} = \frac{\sqrt{3}}{2} \Rightarrow \cos A = \frac{\sqrt{3}}{2}.$$

So we get $A = \pi/6$.   What then is the next?

Next, using the cosine rule again, we get

$$b^2 = c^2 + a^2 - 2ca\cos B \Rightarrow 12 = (3+\sqrt{3})^2 + 6 - 2\cdot\sqrt{6}(3+\sqrt{3})\cos B$$

$$= 6 + 9 + 6\sqrt{3} + 3 - 2(3\sqrt{6} + 3\sqrt{2})\cos B = 18 + 6\sqrt{3} - 6\sqrt{2}(\sqrt{3}+1)\cos B$$

$$\Rightarrow \cos B = \frac{6+6\sqrt{3}}{6\sqrt{2}(\sqrt{3}+1)} = \frac{6(\sqrt{3}+1)}{6\sqrt{2}(\sqrt{3}+1)} = \frac{1}{\sqrt{2}} \Rightarrow \cos B = \frac{1}{\sqrt{2}}.$$

So we get $B = \pi/4$.   How then can we get C?

We have $A + B + C = \pi$, since A, B, and C are the three angles in the triangle ABC, and the sum of the three angles in a triangle is π.

So we get $C = \pi - A - B = \pi - \pi/6 - \pi/4 = 7\pi/12$.