Examples 3 in The Cosine Rule

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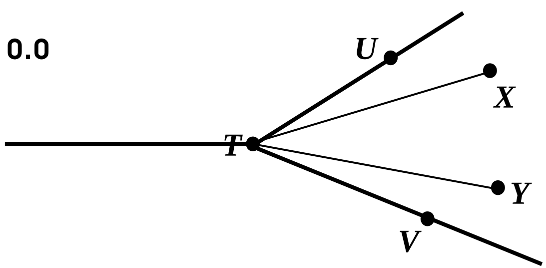
0. A road gets split into two branches making an angle of $5\pi/12$. And there are going to be two gas stations ***U*** and ***V***, each of which will be located in each of the branch roads.

And as shown in the figure below, there are two towns ***X*** and ***Y***, each of which is 30 Km away from the junction ***T*** where the branches begin. The angle ***XTY*** is $\pi/6$, and the angle ***YTV*** is $\pi/12$.

Find the minimum of the sum of the three distances of ***XU***, ***UV***, and ***VY***, that is, the minimum of ***XU + UV + VY***.

Fig. 0.0



1. Suppose again, a road gets split into two branches making an angle of $5\pi/12$. And there are going to be two gas stations ***U*** and ***V***, each of which will be located in each of the branch roads. And a truck depot ***X*** is located between the branches, and is 30 Km away from the junction ***T*** where the branch roads begin.

Find the minimum distance of (***XU + UV+ VX***).

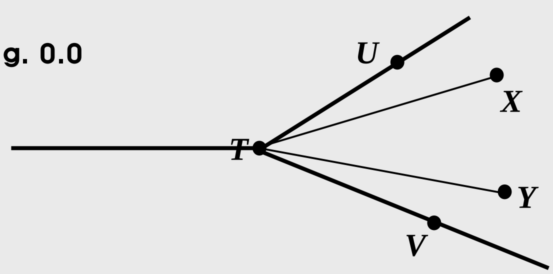
Suggestions or Solutions

To the Problem in the Example 0

A road gets split into two branches making an angle of 5π/12. And there are going to be two gas stations *U* and *V*, each of which will be located in each branch road. And as shown in the figure below, there are two towns *X* and *Y*, each of which is 30 Km away from the junction *T* where the branches begin. The angle *XTY* is π/6, and the angle *YTV* is π/12.

Find the minimum of the sum of the three distances, *XU*, *UV*, and *VY*, that is, the minimum of *XU* + *UV* + *VY*.

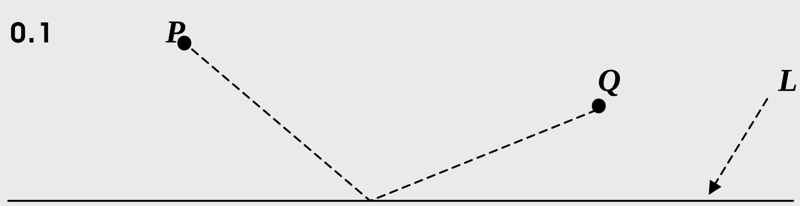
Fig. 0.0



We only know that the two gas stations will be somewhere in the branch roads, but we don't know yet where exactly the gas stations will be. So the exact positions of the two gas stations have not been determined yet. And in fact, we want to determine the positions so that the sum of the distances specified in the problem is the minimum.

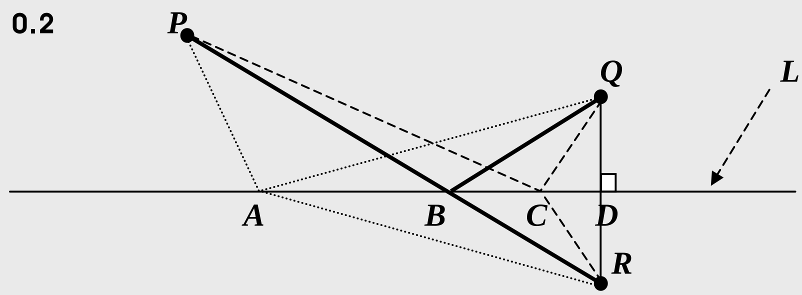
Suppose to begin with, we want to go from ***Q*** to ***P***, but have to visit a point in the line ***L*** below. What point then, do we have to visit in the line ***L*** if the total distance we travel is the minimum?

Fig. 0.1



Assuming the point is ***M***, we can find ***M*** using the triangle inequality. How?

Fig. 0.2



Suppose we put a point ***R*** as shown in the figure above, so that ***QD* = *DR***, and ***QR*** is perpendicular to the line ***L***.

Then, we get ***QB* = *RB***, ***QA* = *RA***, and ***QC* = *RC*** because Δ***QBR***, Δ***QAR***, and Δ***QCR*** are isosceles triangles.

So first, using the triangle inequality, we can get ***QA* + *AP* > *RP***.   How?

That's because ***QA* + *AP* = *RA* + *AP* > *RP*** due to the triangle inequality.
And next, using the triangle inequality again, we can get ***QC* + *CP* > *RP***.   How?

That's because ***QC* + *CP* = *RC* + *CP* > *RP*** due to the triangle inequality.

Now, we know ***A*** and ***C*** are arbitrary points in the line ***L***.
So no matter what the two points ***A*** and ***C*** may be, we get

QA* + *AP* > *RP, and ***QC* + *CP* > *RP***.   So what?

The length of ***RP*** is the minimum distance we can travel.
So the sum, ***QB* + *BP*** is the minimum, that is, ***B*** is the point ***M***.   How?

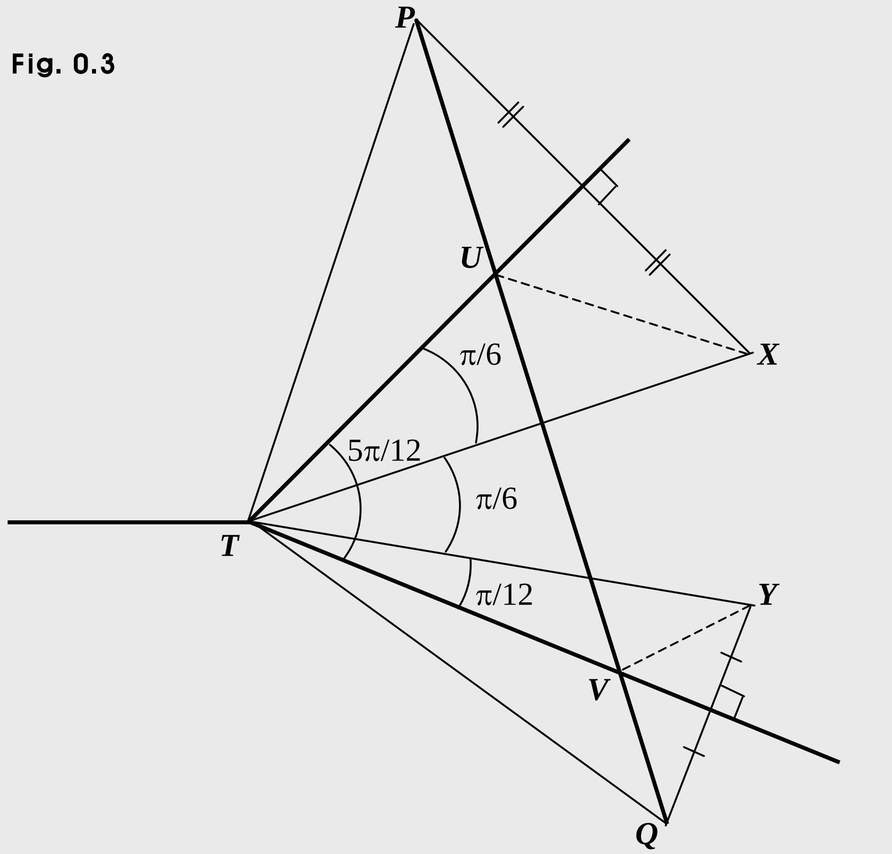
That's because ***QB* + *BP* = *RB* + *BP* = *RP***, since Δ***QBR*** is an isosceles triangle.

Thus, the point ***M*** is the point where the line ***L*** meets a line that connects the destination point ***P*** and the point symmetric to the start point ***Q*** about the line ***L***. And in this case, the symmetric point is the point ***R***.

So we can now use the fact above solving the problem where we want to get the minimum of the sum of the three distances, that is, the minimum of ***XU* + *UV* + *VY***.

Thus, putting into the figure given, the angles given, together with some labels necessary, we can get this:

Fig. 0.3



So finding the length of ***PQ***, we get the minimum of ***XU* + *UV* + *VY***.

Now, to begin with, we have ***TX* = *TY* = 30**. So we get ***PT* = *TQ* = 30**, too.

That's because **Δ*PTX*** and **Δ*YTQ*** are isosceles triangles.
Next, the angle ***PTU*** is π/6, and the angle ***QTV*** is π/12.

That's also because **Δ*PTX*** and **Δ*YTQ*** are isosceles triangles.

So assuming ***A*** is the angle ***PTQ***, we get ***A* = 5π/12 + π/12 + π/6 = 8π/12 = 2π/3**.

How then can we get the length of $\boldsymbol{PQ}$?

We have the cosine rule, where $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$, where $\boldsymbol{A}$ is the angle facing the side $\boldsymbol{a}$, and is the angle between the two sides $\boldsymbol{b}$ and $\boldsymbol{c}$.

So assuming $\boldsymbol{a = PQ}$, $\boldsymbol{b = PT}$, and $\boldsymbol{c = TQ}$, we can say that $\boldsymbol{A = 2\pi/3}$.

And we know $\boldsymbol{\cos(\pi - \theta) = -\cos\theta}$, and $\boldsymbol{\cos\pi/3 = 1/2}$. So we can get

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos 2\pi/3 \Rightarrow a^2 = 30^2 + 30^2 - 2\cdot30\cdot30\,(-\cos\pi/3) = 3\cdot30^2}.$$

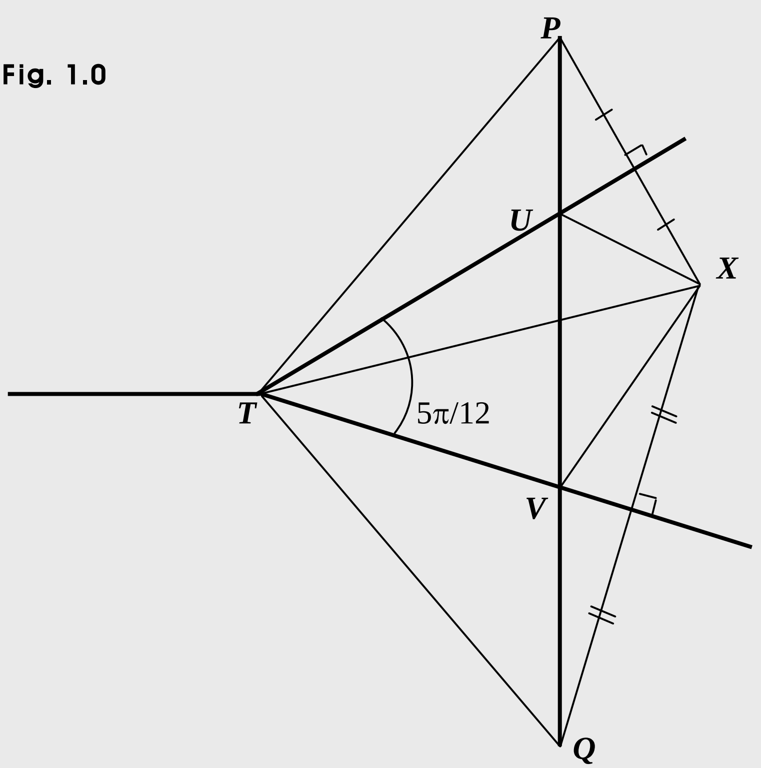
Thus, the minimum is $\boldsymbol{30\sqrt{3}}$.

Suggestions or Solutions
To the **Problem** in the Example 1

Suppose again, a road gets split into two branches making an angle of $5\pi/12$. And there are going to be two gas stations *U* and *V*, each of which will be located in each of the branch roads. And a truck depot *X* is located between the branches, and is 30 Km away from the junction *T* where the branch roads begin. Find the minimum distance of *XU* + *UV*+ *VX*.

Using the fact about the point ***M*** stated in the example 9, we can show the way below, the route that takes the minimum distance.

Fig. 1.0



So finding the length of ***PQ***, we get the minimum of ***XU* + *UV* + *VX***.

Now, to begin with, we have ***TX* = 30**.   So we get ***PT* = *TQ* = 30**, too.

That's because ΔPTX and ΔQTX are isosceles triangles.

Next, the angle PTQ is $10\pi/12$, which is $5\pi/6$. How?

That's also because ΔPTX and ΔYTQ are isosceles triangles.

The sum of the angle UTX and the angle VTX is the same as the sum of the angle QTV and the angle PTU.

So assuming A is the angle PTQ, we get $\mathbf{A = 2 \cdot 5\pi/12 = 10\pi/12 = 5\pi/6}$.

How then can we get the length of PQ?

We have the cosine rule, where $a^2 = b^2 + c^2 - 2bc \cos A$, where A is the angle facing the side a, and is the angle between the two sides b and c.

So assuming $a = PQ$, $b = PT$, and $c = TQ$, we can say that $A = 5\pi/6$.

And we know $\cos(\pi - \theta) = -\cos\theta$, and $\cos \pi/6 = \frac{\sqrt{3}}{2}$.

So we can get

$$a^2 = b^2 + c^2 - 2bc \cos 5\pi/6 \Rightarrow a^2 = 30^2 + 30^2 - 2 \cdot 30 \cdot 30 \, (-\cos \pi/6) = 2 \cdot 30^2 (1 + \tfrac{\sqrt{3}}{2})$$

$$= 30^2 (2 + \sqrt{3}).$$

Thus, the minimum is $30\sqrt{2+\sqrt{3}}$.