Examples in Areas of Triangles

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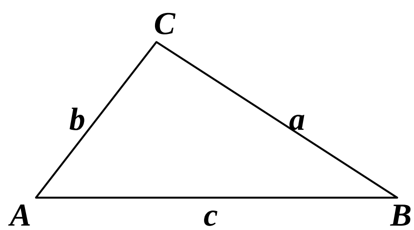
Examples in Areas of Triangles

Doing the examples 0 – 2, refer to the triangle ***ABC*** below.

Fig. 0



0. Find the area of a triangle ***ABC*** where $\boldsymbol{A = \pi/3}$, $\boldsymbol{C = \pi/4}$, and $\boldsymbol{c = 10}$.

1. Suppose in a triangle ***ABC*** where $\boldsymbol{a = 2\sqrt{17}}$, $\boldsymbol{b = 8}$, and $\boldsymbol{c = 10}$, a point ***U*** is in the side ***c***, another point ***V*** is in the side ***b***, the length of the line segment ***AU*** is ***x***, and the length of the line segment ***AV*** is ***y***.

1.0. What then is the value of the product ***xy*** if the area of the triangle ***AUV*** is half the area of the triangle ***ABC***?

1.1. What is the minimum length of the line segment ***UV*** if the area of ***AUV*** is half the area of ***ABC***?

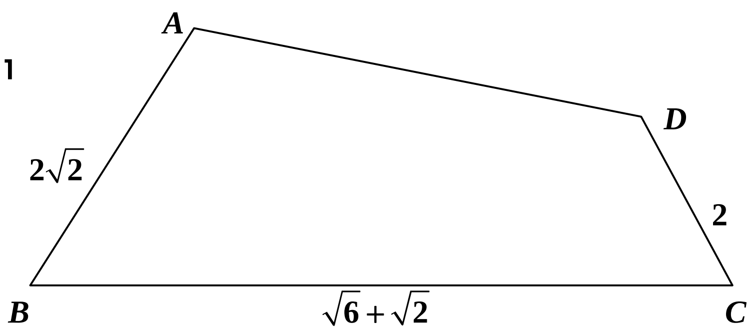
2. Suppose $\boldsymbol{S}$ is the area of a triangle $\boldsymbol{ABC}$, $\boldsymbol{R}$ is the radius of its circumcircle, and $\boldsymbol{r}$ is the radius of its inscribing circle.

2.0. Show that $\boldsymbol{r = \dfrac{2S}{a+b+c}}$, $\mathbf{4}\boldsymbol{SR = abc}$, and $\boldsymbol{S = 2R^2 (\sin A)(\sin B)(\sin C)}$.

2.1. Find $\boldsymbol{r}$ and $\boldsymbol{R}$ assuming $\boldsymbol{a = 13}$, $\boldsymbol{b = 14}$, and $\boldsymbol{c = 15}$.

3. Find the area of a tetragon $\boldsymbol{ABCD}$ below assuming $\boldsymbol{B = \pi/3}$, and $\boldsymbol{C = 5\pi/12}$.

Fig. 1

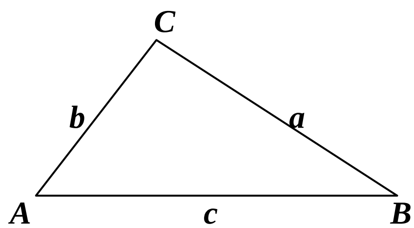


Suggestions or Solutions

To the Problem in the Example 0

Find the area of a triangle ABC where $A = \pi/3$, $C = \pi/4$, and $c = 10$.

Fig. 0.0



Assuming $\boldsymbol{d}$ is the distance from the point $\boldsymbol{B}$ to $\boldsymbol{b}$, we get $\boldsymbol{d = 10 \sin 60^{o} = 5\sqrt{3}}$.

So assuming $\boldsymbol{s}$ is the area, and $\boldsymbol{b}$ is the base, we get $\boldsymbol{s = \frac{5b\sqrt{3}}{2}}$.

And we can put $\boldsymbol{b}$ this way: $\boldsymbol{10 \cos 60^{o} + a \cdot \cos 45^{o}}$. And getting $\boldsymbol{a}$, we get

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow a = \frac{c}{\sin C} \cdot \sin A = \frac{10}{\sqrt{2}/2} \cdot \frac{\sqrt{3}}{2} = \frac{20}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{10\sqrt{3}}{\sqrt{2}} = \frac{10\sqrt{6}}{2} = 5\sqrt{6}.$$

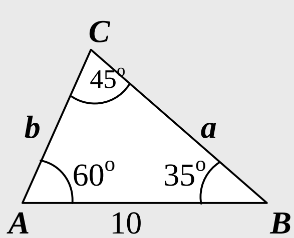
So we get $\boldsymbol{b = 10 \cos 60^{o} + a \cdot \cos 45^{o}}$

$$= 10 \cdot \tfrac{1}{2} + a\tfrac{\sqrt{2}}{2} = \tfrac{10 + a\sqrt{2}}{2} = \tfrac{10 + 5\sqrt{6}\sqrt{2}}{2} = \tfrac{10 + 10\sqrt{3}}{2} = 5(1 + \sqrt{3}).$$

Thus, we get $\boldsymbol{s = \frac{5b\sqrt{3}}{2} = \frac{25(1+\sqrt{3})\sqrt{3}}{2} = \frac{25(3+\sqrt{3})}{2}}$.

If not quite sure of the idea behind the processes above, follow the steps below.
To begin with, using the information given, we can put the triangle the way below.

Fig. 0.1



Basically, the area of a triangle is the base times the height over two.
And we want to determine the base first.

So for instance, in this example, finding the side $\boldsymbol{b}$, and the height, we can get the area.
What is the height though?

We can get the distance from the point at $\boldsymbol{B}$ to the side $\boldsymbol{b}$.

Then, the distance is the height since $\boldsymbol{b}$ has been chosen to be the base.

And the distance is in fact $\mathbf{10 \sin 60^o}$, which is $\mathbf{5\sqrt{3}}$ since $\mathbf{\sin 60^o = \frac{\sqrt{3}}{2}}$.

So assuming $\boldsymbol{b}$ is the base, we can get $\frac{1}{2}\boldsymbol{b}\cdot 5\sqrt{3}$, which is $\frac{5\boldsymbol{b}\sqrt{3}}{2}$, which is the area.
How then can we get the side $\boldsymbol{b}$?

We have a tool called the sine rule, where $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where $\boldsymbol{R}$ is the radius of the circumcircle to the triangle $\boldsymbol{ABC}$.

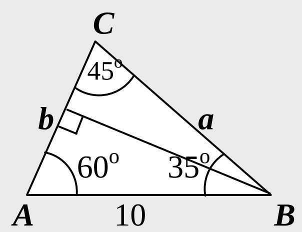
So the circle passes through all the three vertices of the triangle $\boldsymbol{ABC}$.
And we know the three angles, and the value of $\boldsymbol{c}$. So?

So using $\frac{b}{\sin B} = \frac{c}{\sin C}$, we can get $\boldsymbol{b}$.

The angle $\boldsymbol{B}$ is however $\mathbf{35^o}$, the sine of which cannot be readily found unless we use a calculator. What else then can we do to get the side $\boldsymbol{b}$?

We can readily get $\mathbf{\cos 60^o}$ and $\mathbf{\cos 45^o}$. So we can put $\boldsymbol{b}$ this way: $\mathbf{10 \cos 60^o} + \boldsymbol{a}\cdot\mathbf{\cos 45^o}$.
How?

Fig. 0.2



So we can set **$b = 10\cos 60^o + a \cdot \cos 45^o$**. What then about a?

We know **$\sin 60^o = \frac{\sqrt{3}}{2}$**, and **$\sin 45^o = \frac{\sqrt{2}}{2}$**. So?

So we can use the sine rule to get a. Thus, using the rule, we get

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow a = \frac{c}{\sin C}\cdot \sin A = \frac{10}{\frac{\sqrt{2}}{2}}\cdot\frac{\sqrt{3}}{2} = \frac{20}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{10\sqrt{3}}{\sqrt{2}} = \frac{10\sqrt{6}}{2} = 5\sqrt{6}.$$ So we get $a = 5\sqrt{6}$.

So putting it into this: **$b = 10\cos 60^o + a \cdot \cos 45^o$**, where **$\cos 60^o = 1/2$**, and **$\cos 45^o = \frac{\sqrt{2}}{2}$**,

we get $b = \frac{10}{2} + \frac{a\sqrt{2}}{2} = \frac{10 + a\sqrt{2}}{2} = \frac{10 + 5\sqrt{6}\sqrt{2}}{2} = \frac{10 + 10\sqrt{3}}{2} = 5(1+\sqrt{3})$.

And we know that the area is $\frac{5b\sqrt{3}}{2}$.

So assuming s is the area, we get $s = \frac{5b\sqrt{3}}{2} = \frac{25(1+\sqrt{3})\sqrt{3}}{2} = \frac{25(3+\sqrt{3})}{2}$.

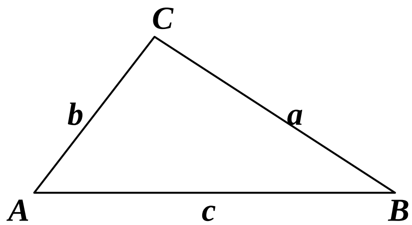
Suggestions or Solutions

To the Problem 0 in the Example 1

Suppose in a triangle *ABC* where $a = 2\sqrt{17}$, $b = 8$, and $c = 10$, a point *U* is in the side *c*, another point *V* is in the side *b*, the length of the line segment *AU* is *x*, and the length of the line segment *AV* is *y*.

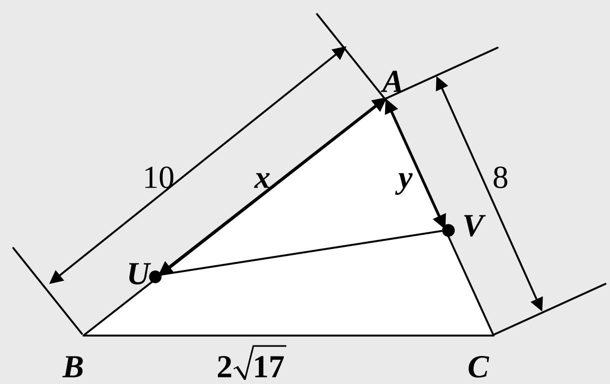
What then is the value of the product *xy* if the area of the triangle *AUV* is half the area of the triangle *ABC*?

Fig. 1.0.0



To begin with, using the information given, we can put the triangle the way below.

Fig. 1.0.1



Assuming ***T*** is the area of Δ***AUV***, and ***t*** is the area of Δ***ABC***, we want determine the value of the product ***xy*** so that we get ***T*** =2***t***.   What then about the value of ***xy***?

In fact, finding the area of Δ***AUV***, we get to see the value of ***xy***.

That's because the area will have the value of ***xy***. So let's now, find the area of Δ***AUV***.

Assuming x is the base, we can put the area of ΔAUV this way $t = \dfrac{xy \sin A}{2}$.

And assuming the side b is the base, we can put the area of ΔABC the way below.

$T = \dfrac{bc \sin A}{2}$ where $b = 8$, and $c = 10$.

So we get $T = 40 \sin A$.   How then can we get the $\sin A$?

We don't really need it.   Why not?

What we want is xy.   So we can use this instead: $T = 2t$.

Thus, using it, we get

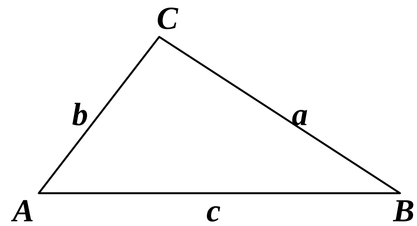
$$T = 2t \Rightarrow 40 \sin A = \frac{2xy \sin A}{2} = xy \sin A \Rightarrow 40 \sin A = xy \sin A \Rightarrow xy = 40.$$

Suggestions or Solutions
To the Problem 1 in the Example 1

Suppose in a triangle *ABC* where $a = 2\sqrt{17}$, $b = 8$, and $c = 10$, a point *U* is in the side *c*, another point *V* is in the side *b*, the length of the line segment *AU* is *x*, and the length of the line segment *AV* is *y*.

What then is the minimum length of the line segment *UV* if the area of *AUV* is half the area of *ABC*?

Fig. 1.1.0



Setting $s = UV$, we can get $s^2 = x^2 + y^2 - 2xy\cos A$.

And we get $\cos A = (b^2 + c^2 - a^2)/(2bc) = (64 + 100 - 68)/160 = 96/160 = 3/5$.

Also, we have $xy = 40$. So we can set $y = 40/x$.

Thus, we get $s^2 = x^2 + y^2 - 2xy\cos A = x^2 + (40/x)^2 - 2\cdot 40(3/5) = x^2 + 40^2/x^2 - 48$.

So assuming $u = x^2$, and $v = s^2$, we can set $v = u + 40^2/u - 48$ for $u > 0$.

And assuming, $h(u) = u + \dfrac{40^2}{u}$, $f(u) = u$, and $g(u) = \dfrac{40^2}{u}$, we can set

$v = h(u) = f(u) + g(u)$ for $u > 0$.

And referring to the graph above, we can see that when $u = \dfrac{40^2}{u}$, h gets its minimum

value. And we get $u = \dfrac{40^2}{u} \Rightarrow u^2 = 40^2 \Rightarrow u = 40$.

Thus, when $u = 40$, v gets its minimum, which is $40 + \dfrac{40^2}{40} - 48 = 80 - 48 = 32$.

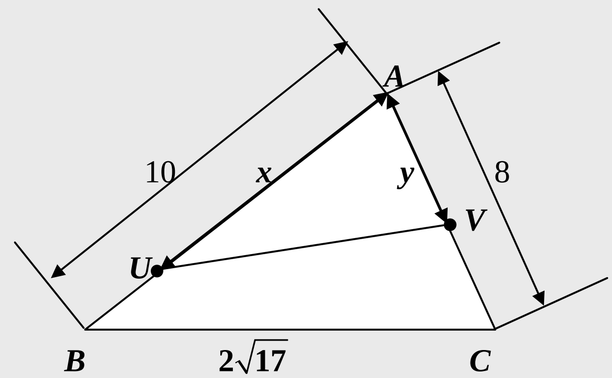
And we know $v = s^2$, which is the square of the length of UV.

So the minimum length of UV is $\sqrt{32} = 4\sqrt{2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, using the information given, we can put the triangle the way below.

Fig. 1.1.1



It can be the case, of course, $\boldsymbol{x}$ and $\boldsymbol{y}$ can change keeping $\boldsymbol{xy}$ constant to be 40.

That is to say that $\boldsymbol{U}$ and $\boldsymbol{V}$ can move along the sides respectively so that the length of the line segment $\boldsymbol{UV}$ can change keeping the area of $\Delta\boldsymbol{AUV}$ half the area of $\Delta\boldsymbol{ABC}$, that is, keeping unchanged the equation $\boldsymbol{T = 2t}$.

So we may want to keep track of the values of the line segment $\boldsymbol{UV}$ as other values vary.

Thus, we may want to take the length of $\boldsymbol{UV}$ as a variable.

So for instance, we can set $\boldsymbol{s = UV}$.
And we know that $\boldsymbol{x}$ and $\boldsymbol{y}$ vary keeping $\boldsymbol{xy}$ 40 though, of course.

How then can we relate $\boldsymbol{s}$, $\boldsymbol{x}$, and $\boldsymbol{y}$?

We have the cosine rule, where $\boldsymbol{p^2 = q^2 + r^2 - 2qr\cos P}$ for $\Delta\boldsymbol{PQR}$.

So we can set $\boldsymbol{s^2 = x^2 + y^2 - 2xy\cos A}$ for $\Delta\boldsymbol{AUV}$.   How then can we get $\boldsymbol{\cos A}$?

We have this, too: $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$, where $\boldsymbol{a = 2\sqrt{17}}$, $\boldsymbol{b = 8}$, and $\boldsymbol{c = 10}$.

So we can get $\boldsymbol{\cos A}$ this way:

$\boldsymbol{\cos A = (b^2 + c^2 - a^2)/(2bc) = (64 + 100 - 68)/160 = 96/160 = 3/5}$.

So we get $\boldsymbol{s^2 = x^2 + y^2 - 2xy\cos A = x^2 + y^2 - 2xy(3/5) = x^2 + y^2 - 6xy/5}$.

Thus, we get $\boldsymbol{s^2 = x^2 + y^2} - \dfrac{6xy}{5}$. What then is the next?

We have this, too: $\boldsymbol{xy = 40}$, which is the main key to this problem.

So we can get $\boldsymbol{y = 40/x}$.

Then, we get $\boldsymbol{s^2 = x^2 + y^2} - \dfrac{6xy}{5} = \boldsymbol{x^2 + 40^2/x^2} - \dfrac{6}{5}x\dfrac{40}{\boldsymbol{x}} = \boldsymbol{x^2 + 40^2/x^2 - 48}$.

That is, we get $\boldsymbol{s^2 = x^2 + 40^2/x^2 - 48}$. And we know $\boldsymbol{x^2}$ is always positive.

So assuming $\boldsymbol{u = x^2}$, and $\boldsymbol{v = s^2}$, we can set $\boldsymbol{v = u + 40^2/u - 48}$ for $\boldsymbol{u > 0}$.

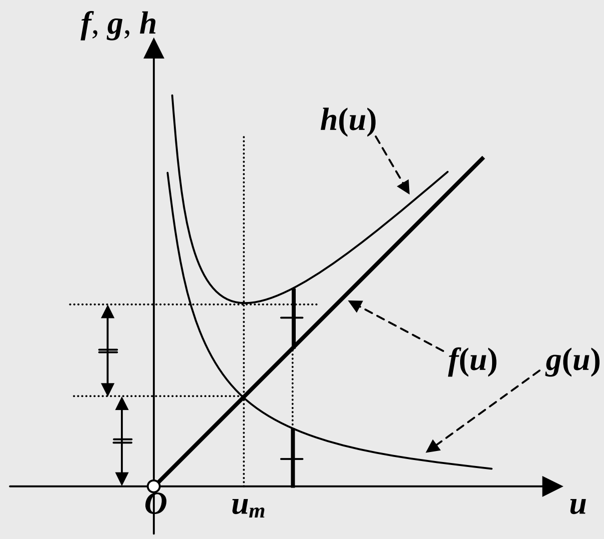
And thus, getting the minimum of $\boldsymbol{v}$, we can get the minimum of $\boldsymbol{s}$, which is the minimum of $\boldsymbol{UV}$, of course. How then can we get the minimum of $\boldsymbol{v}$?

We can notice that even if $\boldsymbol{u}$ changes, 48 remains 48, and thus, remains constant.
So we can keep track of $\boldsymbol{u + 40^2/u}$ to find the minimum of $\boldsymbol{v}$.

And assuming, $\boldsymbol{h(u) = u + 40^2/u}$, $\boldsymbol{f(u) = u}$, and $\boldsymbol{g(u) = 40^2/u}$, we can set

$\boldsymbol{v = h(u) = f(u) + g(u)}$ for $\boldsymbol{u > 0}$. And putting it in a graph, we get this:

Fig. 1.1.2



So we can see that when $\boldsymbol{f(u) = g(u)}$, that is, $\boldsymbol{u = 40^2/u}$, $\boldsymbol{h}$ gets its minimum value.
That is, if $\boldsymbol{u = u_m}$, $\boldsymbol{h}$ gets its minimum.

Thus, finding $\boldsymbol{u_m}$ in the case above, we get $\boldsymbol{u_m = 40^2/u_m \Rightarrow u_m^2 = 40^2 \Rightarrow u_m = 40}$.
So when $\boldsymbol{u = 40}$, $\boldsymbol{h}$ gets its minimum.

And we know $\boldsymbol{v = u + 40^2/u - 48}$, and $\boldsymbol{h(u) = u + 40^2/u}$.
So we can set $\boldsymbol{v = h(u) - 48}$, where 48 is not subject to $\boldsymbol{u}$.

Thus, when $\boldsymbol{u = 40}$, $\boldsymbol{v}$ gets its minimum, which is $\boldsymbol{40 + 40^2/40 - 48 = 80 - 48 = 32}$.
That is, the minimum of $\boldsymbol{v}$ is 32.

And we know $\boldsymbol{v = s^2}$, which is the square of the length of $\boldsymbol{UV}$.
So the minimum length of $\boldsymbol{UV}$ is $\boldsymbol{\sqrt{32} = 4\sqrt{2}}$.

By the way, what are $\boldsymbol{x}$ and $\boldsymbol{y}$ when $\boldsymbol{UV}$ gets its minimum?

We know that we set $\boldsymbol{u = x^2}$, and that when $\boldsymbol{u = 40}$, $\boldsymbol{UV}$ gets its minimum.
So when $\boldsymbol{UV}$ gets its minimum, we get $\boldsymbol{x^2 = 40 \Rightarrow x = \sqrt{40} = 2\sqrt{10}}$.

And we have $xy = 40$. So when $x = 2\sqrt{10}$, we get $y = 40/x = 40/2\sqrt{10} = 2\sqrt{10}$.

And we can get the same the way below, too:

We have $s^2 = x^2 + y^2 - 6xy/5$, where s is the length of UV.

And we can get $s^2 = x^2 + y^2 - 6xy/5 = x^2 + y^2 - 2xy + 2xy - 6xy/5 = (x - y)^2 + 4xy/5$.

Also, we know that $xy = 40$. So we get $s^2 = (x - y)^2 + 32$.

Thus, when $x = y$, s^2 gets its minimum, which is 32.

And we know s^2 is the square of the length of UV.

So when $x = y$, the length of UV gets its minimum, which is $\sqrt{32} = 4\sqrt{2}$.

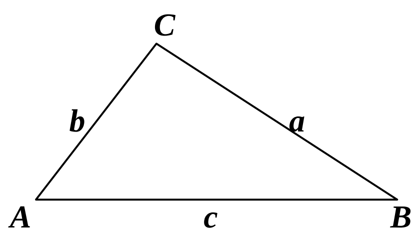
Suggestions or Solutions

To the Problem 0 in the Example 2

Assuming S is the area of a triangle ABC, R is the radius of its circumcircle, and r is the radius of its inscribing circle, show that

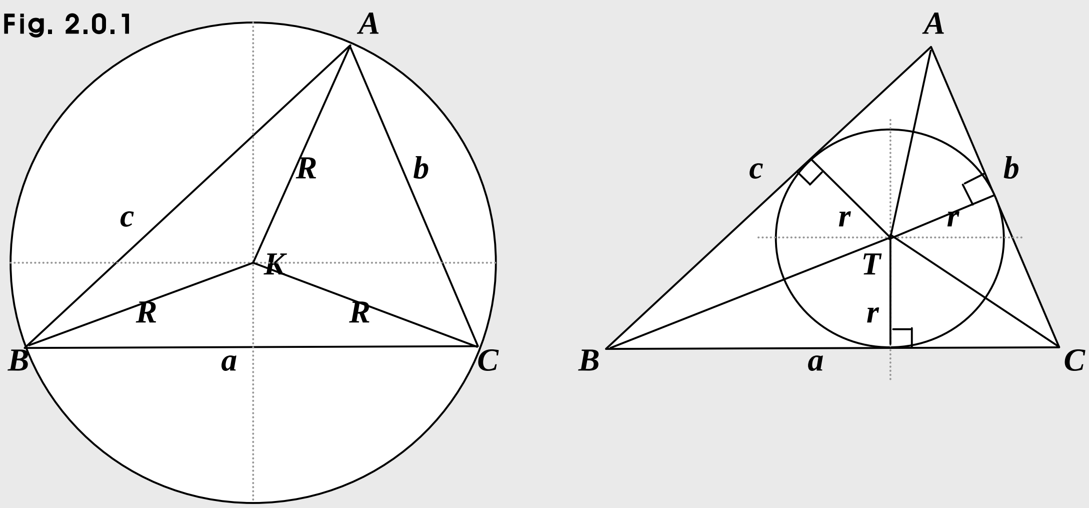
$r = \frac{2S}{a+b+c}$, **$4SR = abc$, and $S = 2R^2 (\sin A)(\sin B)(\sin C)$.**

Fig. 2.0.0



To begin with, using the information given, we can put the triangle the way below.

Fig. 2.0.1



Then first, what do we mean by $r = \frac{2S}{a+b+c}$?

We can put it this way, too: $S = \frac{(a+b+c)r}{2}$, where r is the radius of the circle inscribing to the triangle $\boldsymbol{ABC}$. What do we mean by though, the circle inscribing?

It is the circle to which the three sides of the triangle $\boldsymbol{ABC}$ are tangent.

So we can say that the inscribing circle is tangent to all the three sides of the triangle.

And if a circle is tangent to all the sides of a polygon as a square, the circle can be said to inscribe the polygon. Not every polygon has its inscribing circle though. A rectangle cannot have it. Every triangle though, has its inscribing circle.

Now, the equation above is in fact, quite straightforward, because $\boldsymbol{r}$ is the height of each of the three triangles, $\Delta \boldsymbol{TAB}$, $\Delta \boldsymbol{TBC}$, and $\Delta \boldsymbol{TCA}$ in the figure above.
Assuming for instance, $\boldsymbol{a}$ is the base of $\Delta \boldsymbol{TBC}$, and $\boldsymbol{S_a}$ is the area of $\Delta \boldsymbol{TBC}$, we can set $\boldsymbol{S_a} = \frac{\boldsymbol{ar}}{2}$. And the same is true, also, for the other two triangles, $\Delta \boldsymbol{TAB}$, and $\Delta \boldsymbol{TCA}$.

So we get $\boldsymbol{S_b} = \frac{\boldsymbol{br}}{2}$, and $\boldsymbol{S_c} = \frac{\boldsymbol{cr}}{2}$. Thus, assuming $\boldsymbol{S}$ is the area of $\Delta \boldsymbol{ABC}$, we get

$$\boldsymbol{S = S_a + S_b + S_c} = \frac{\boldsymbol{ar}}{2} + \frac{\boldsymbol{br}}{2} + \frac{\boldsymbol{cr}}{2} = \boldsymbol{(a + b + c)}\frac{\boldsymbol{r}}{2} \Rightarrow \boldsymbol{S} = \frac{\boldsymbol{(a+b+c)r}}{2} \Rightarrow \boldsymbol{r} = \frac{\boldsymbol{2S}}{\boldsymbol{a+b+c}}.$$

So knowing the radius of the inscribing circle, along with all the three sides, we can get the area of the triangle.

Let's next, move on to $\boldsymbol{4SR = abc}$, where $\boldsymbol{R}$ is the radius of the circumcircle, which passes through all the three vertices of the triangle.

And if a circle passes through all the vertices of a polygon as a pentagon, the circle can be said to circumscribe the polygon. Not every polygon however, has its circumscribing circle, called circumcircle, for short. For instance, some quadrangles (tetragons) and some pentagons cannot have it.
Now, what do we mean by $\boldsymbol{4SR = abc}$?

We can put it this way, too: $\boldsymbol{S} = \frac{\boldsymbol{abc}}{\boldsymbol{4R}}$, which is the area of the triangle $\boldsymbol{ABC}$.
And of course, we can put the area in another way, too.
Knowing two sides and the angle between the two sides, we can get the area.
For instance, if we know $\boldsymbol{b}$ and $\boldsymbol{c}$, together with the angle $\boldsymbol{A}$, we can set $\boldsymbol{S} = \frac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$.
What then about $\boldsymbol{R}$, which is the radius of the circumcircle?

What equation has such a radius $\boldsymbol{R}$, the radius of the circumcircle?

We have the sine rule, where $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

So using the sine rule, we can get $\frac{a}{\sin A} = 2R \Rightarrow \sin A = \frac{a}{2R}$.

Thus, we get $S = \frac{bc \sin A}{2} = \frac{abc}{4R} \Rightarrow 4SR = abc$.
So knowing the radius of the circumcircle, along with all the three sides, we can get the area of the triangle, too.

And let's next, move on to the last case where $S = 2R^2 (\sin A)(\sin B)(\sin C)$.
We know S is the area of the triangle ABC.

So the equation where $S = 2R^2 (\sin A)(\sin B)(\sin C)$ is just another expression for the area of the triangle ABC.
And the area is put in terms of the radius of the circumcircle and the all the three angles of the triangle ABC. How then can we get the equation?

The equation is in terms of the sines of all the three angles.
So we can think of the sine rule. And we have $4SR = abc$.

So first, putting all the sides in terms of the sines, we get
$a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$.

So next, putting all the above into the equation where $4SR = abc$, we get

$$4SR = abc = (2R \sin A)(2R \sin B)(2R \sin C) = 8R^3 (\sin A)(\sin B)(\sin C)$$
$$\Rightarrow 4SR = 8R^3 (\sin A)(\sin B)(\sin C) \Rightarrow S = 2R^2 (\sin A)(\sin B)(\sin C).$$

And we can get the same the way below, too:

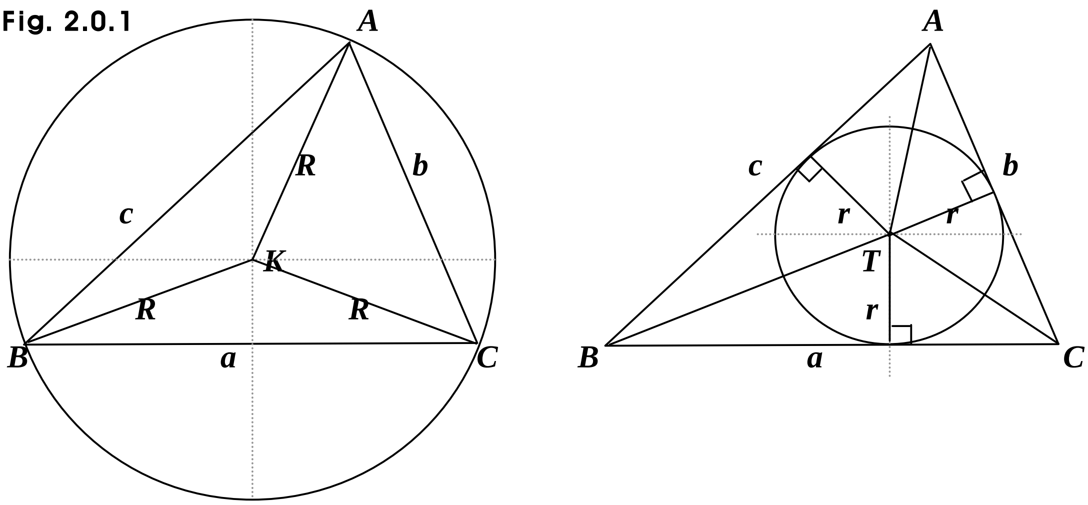
We can put S this way, too: $S = (bc \sin A)/2$.

And using the sine rule again, we get $b = 2R \sin B$ and $c = 2R \sin C$.

So we get $S = \frac{bc \sin A}{2} = (2R \sin B)(2R \sin C)(\sin A)\frac{1}{2} = 2R^2 (\sin A)(\sin B)(\sin C)$.

In short:

Fig. 2.0.1



Referring to the figure above, we can get

$$\boldsymbol{S} = \frac{\boldsymbol{ar}}{2} + \frac{\boldsymbol{br}}{2} + \frac{\boldsymbol{cr}}{2} = \boldsymbol{(a + b + c)}\frac{\boldsymbol{r}}{2} \Rightarrow \boldsymbol{S} = \frac{\boldsymbol{(a+b+c)r}}{\boldsymbol{2}} \Rightarrow \boldsymbol{r} = \frac{\boldsymbol{2S}}{\boldsymbol{a+b+c}}.$$

Next, we can set $\boldsymbol{S} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$. And using the sine rule, we can get

$$\frac{\boldsymbol{a}}{\mathbf{sin}\ \boldsymbol{A}} = \mathbf{2}\boldsymbol{R} \Rightarrow \mathbf{sin}\ \boldsymbol{A} = \frac{\boldsymbol{a}}{\mathbf{2}\boldsymbol{R}}.\quad \text{So we get } \boldsymbol{S} = \frac{\boldsymbol{bc}\sin \boldsymbol{A}}{2} = \frac{\boldsymbol{abc}}{\mathbf{4}\boldsymbol{R}} \Rightarrow \mathbf{4}\boldsymbol{SR} = \boldsymbol{abc}.$$

And next, using the sine rule again, we get $\boldsymbol{a} = \mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{A}$, $\boldsymbol{b} = \mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{B}$, and $\boldsymbol{c} = \mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{C}$.
So next, putting all the above into the equation where $\mathbf{4}\boldsymbol{SR} = \boldsymbol{abc}$, we get

$$\mathbf{4}\boldsymbol{SR} = \boldsymbol{abc} = (\mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{A})(\mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{B})(\mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{C}) = \mathbf{8}\boldsymbol{R}^{\mathbf{3}}\ (\mathbf{sin}\ \boldsymbol{A})(\mathbf{sin}\ \boldsymbol{B})(\mathbf{sin}\ \boldsymbol{C})$$

$$\Rightarrow \mathbf{4}\boldsymbol{SR} = \mathbf{8}\boldsymbol{R}^{\mathbf{3}}\ (\mathbf{sin}\ \boldsymbol{A})(\mathbf{sin}\ \boldsymbol{B})(\mathbf{sin}\ \boldsymbol{C}) \Rightarrow \boldsymbol{S} = \mathbf{2}\boldsymbol{R}^{\mathbf{2}}\ (\mathbf{sin}\ \boldsymbol{A})(\mathbf{sin}\ \boldsymbol{B})(\mathbf{sin}\ \boldsymbol{C}).$$

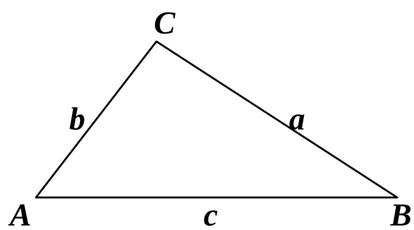
And we can put $\boldsymbol{S}$ this way, too: $\boldsymbol{S} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$.
And using the sine rule again, we get: $\boldsymbol{b} = \mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{B}$ and $\boldsymbol{c} = \mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{C}$.

So we get $\boldsymbol{S} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2} = (\mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{B})(\mathbf{2}\boldsymbol{R}\ \mathbf{sin}\ \boldsymbol{C})(\mathbf{sin}\ \boldsymbol{A})\dfrac{1}{2} = \mathbf{2}\boldsymbol{R}^{\mathbf{2}}\ (\mathbf{sin}\ \boldsymbol{A})(\mathbf{sin}\ \boldsymbol{B})(\mathbf{sin}\ \boldsymbol{C})$.

Suggestions or Solutions
To the Problem 1 in the Example 2

Suppose again, *S* is the area of a triangle *ABC*, *R* is the radius of its circumcircle, and *r* is the radius of its inscribing circle.
What then, are *r* and *R* if *a* = 13, *b* = 14, and *c* = 15?

Fig. 2.1.0



We have $\boldsymbol{S = \sqrt{t(t-a)(t-b)(t-c)}}$ where $\boldsymbol{2t = a + b + c}$.

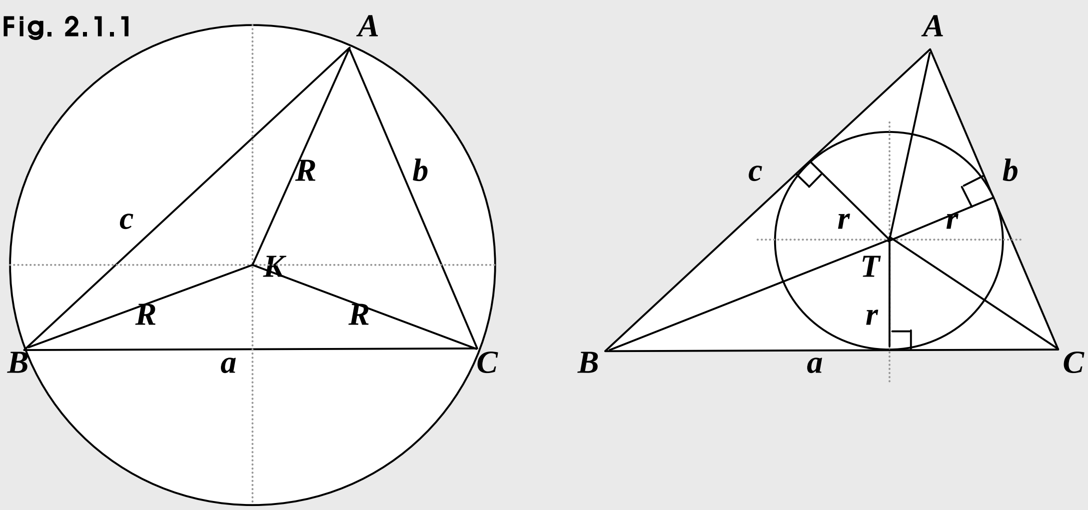
So first, we get $\boldsymbol{2t = a + b + c = 13 + 14 + 15 = 42 \Rightarrow t = 21}$. Thus next, we can get $\boldsymbol{t(t-a)(t-b)(t-b) = 21(21-13)(21-14)(21-15) = 21\cdot8\cdot7\cdot6 = 2^4\cdot3^2\cdot7^2 = 84^2}$.

So we get $\boldsymbol{S = 84}$. And we have $\boldsymbol{r = \frac{2S}{a+b+c}}$ and $\boldsymbol{4SR = abc}$. So we get $\boldsymbol{r = 2\cdot84/42 = 4}$, and $\boldsymbol{R = abc/(4S) = 13\cdot14\cdot15/(4\cdot84) = 13\cdot2\cdot7\cdot3\cdot5/(2\cdot2\cdot7\cdot3\cdot4) = 65/8}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, using the information given, we can put the triangle the way below.

Fig. 2.1.1



We have $\boldsymbol{r} = \frac{2S}{a+b+c}$ and $\boldsymbol{4SR = abc}$, proved in the previous example.

So finding $\boldsymbol{S}$, we can get $\boldsymbol{r}$ and $\boldsymbol{R}$. How then, can we get $\boldsymbol{S}$, i.e., the area of the triangle?

None of the angles are given. So we cannot use the formula $S = \dfrac{bc \sin A}{2}$.
We can find though, the angles using the cosine rule as $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$.
Let's this time however, get the area without using any of the angles.

We have a tool called Heron's formula, which says this:

Assuming $\boldsymbol{2t = a + b + c}$, we get $S = \sqrt{t(t-a)(t-b)(t-c)}$ for ΔABC.

So getting $\boldsymbol{t}$ first, we get $\boldsymbol{2t = a + b + c = 13 + 14 + 15 = 42 \Rightarrow t = 21}$.

Thus next, we can get $\boldsymbol{t(t-a)(t-b)(t-b) = 21(21-13)(21-14)(21-15)}$
$\boldsymbol{= 21 \cdot 8 \cdot 7 \cdot 6 = 2^4 \cdot 3^2 \cdot 7^2 = 84^2}$. So we get $\boldsymbol{S = 84}$.

Now, we have: $\boldsymbol{r} = \frac{2S}{a+b+c}$ and $\boldsymbol{4SR = abc}$.

So we can get first, $\boldsymbol{r = 2 \cdot 84/42 = 4}$. And next, we get

$\boldsymbol{R = abc/(4S) = 13 \cdot 14 \cdot 15/(4 \cdot 84) = 13 \cdot 2 \cdot 7 \cdot 3 \cdot 5/(2 \cdot 2 \cdot 7 \cdot 3 \cdot 4) = 13 \cdot 5/(2 \cdot 4) = 65/8}$.

In short:

We have $S = \sqrt{t(t-a)(t-b)(t-c)}$ where $\boldsymbol{2t = a + b + c}$.

So first, we get $\boldsymbol{2t = a + b + c = 13 + 14 + 15 = 42 \Rightarrow t = 21}$.
Thus next, we can get
$\boldsymbol{t(t-a)(t-b)(t-b) = 21(21-13)(21-14)(21-15) = 21 \cdot 8 \cdot 7 \cdot 6 = 2^4 \cdot 3^2 \cdot 7^2 = 84^2}$.

So we get $\boldsymbol{S = 84}$.

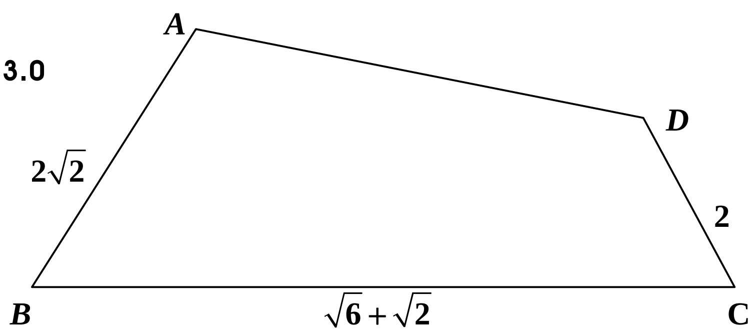
And we have $\boldsymbol{r} = \frac{2S}{a+b+c}$ and $\boldsymbol{4SR = abc}$. So we get

$\boldsymbol{r = 2 \cdot 84/42 = 4}$, and $\boldsymbol{R = abc/4S = 13 \cdot 14 \cdot 15/(4 \cdot 84) = 13 \cdot 2 \cdot 7 \cdot 3 \cdot 5/(2 \cdot 2 \cdot 7 \cdot 3 \cdot 4) = 65/8}$.

Suggestions or Solutions
To the Problem in the Example 3

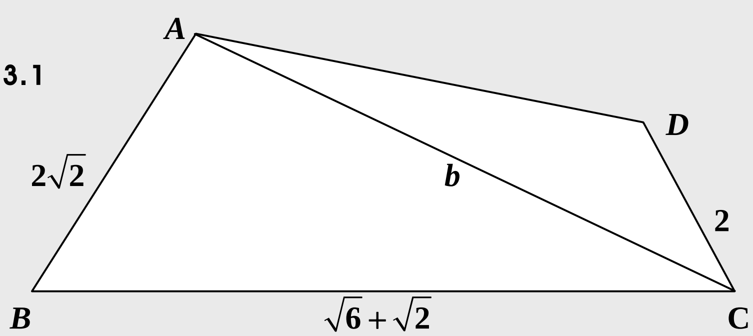
Find the area of a tetragon *ABCD* below assuming $B = \pi/3$, and $C = 5\pi/12$.

Fig. 3.0



A tetragon can be said to be a sum of many triangles, and thus, we can partition the tetragon into two triangles as shown below.

Fig. 3.1



So assuming T is the area of the tetragon, we can put T the way below.

$T = t_0 + t_1$, where t_0 is the area of ΔABC, and t_1 is the area of ΔADC.

How then, can we get the area t_0?

Knowing two sides and the angle between the two, we can get the area.

For instance, assuming S is the area of ΔXYZ, we can get $S = \dfrac{xy \sin Z}{2}$.

We know the angle $\boldsymbol{B}$ and the two sides $\boldsymbol{a}$ and $\boldsymbol{c}$ in ΔABC.

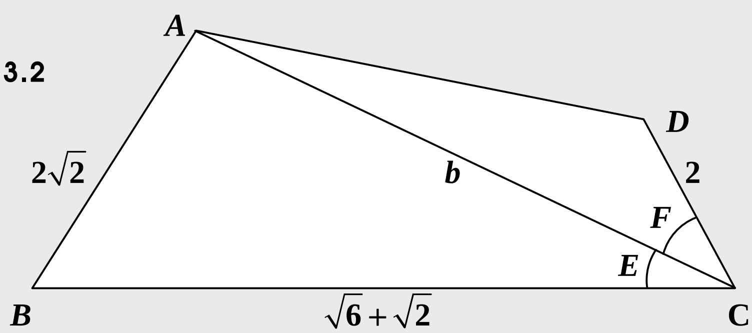
And we have $\boldsymbol{B = \pi/3}$, $\boldsymbol{a = 2\sqrt{2}}$, and $\boldsymbol{c = \sqrt{6}+\sqrt{2}}$. So we can get

$$t_0 = \frac{ca\sin B}{2} = \{(\sqrt{6}+\sqrt{2})2\sqrt{2}\sin 60^o\}\frac{1}{2}$$

$$= (\sqrt{6}+\sqrt{2})2\sqrt{2}\cdot\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{2} = (6+2\sqrt{3})\cdot\tfrac{1}{2} = 3+\sqrt{3}.$$

And the next is t_1, which is the area of ΔADC.

Fig. 3.2



Knowing two sides and the angle between the two, we can get the area.
We don’t know however, the side $\boldsymbol{b}$ and the angle $\boldsymbol{F}$.
We can get $\boldsymbol{b}$ though. How?

We have the cosine rule $\boldsymbol{x^2 = y^2 + z^2 - 2yz\cos X}$ where $\boldsymbol{X}$ is the angle between $\boldsymbol{y}$ an $\boldsymbol{z}$.
So using the rule, we get

$$b^2 = (2\sqrt{2})^2 + (\sqrt{6}+\sqrt{2})^2 - 2(2\sqrt{2})(\sqrt{6}+\sqrt{2})\cdot\tfrac{1}{2}$$

$$= 8 + 8 + 4\sqrt{3} - 4\sqrt{3} - 4 = 12 \Rightarrow b = 2\sqrt{3}.$$

Thus, getting the angle $\boldsymbol{F}$, we can get the area t_1, which is $\boldsymbol{(2b\sin F)/2}$.
How then can we get the angle $\boldsymbol{F}$?

We know in the triangle $\boldsymbol{ABC}$, the angles $\boldsymbol{B}$ and $\boldsymbol{C}$ and the two sides, one is $\boldsymbol{b}$, which is $\mathbf{2\sqrt{3}}$, and the other is $\mathbf{2\sqrt{2}}$ facing the angle $\boldsymbol{E}$. So what?

Since we know the angle $\boldsymbol{C}$, finding the angle $\boldsymbol{E}$, we can get the angle $\boldsymbol{F}$.

And finding the angle $\boldsymbol{E}$, we can use the sine rule $\dfrac{x}{\sin X} = \dfrac{y}{\sin Y} = \dfrac{z}{\sin Z} = 2R$ for ΔXYZ.

So using the rule, we get $\dfrac{b}{\sin B} = \dfrac{2\sqrt{2}}{\sin E} \Rightarrow \sin E = \dfrac{2\sqrt{2}}{b} \cdot \sin B = \dfrac{\sqrt{2}}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{2}}{2}$.

Thus, we get: $E = \pi/4$ since $0 < E < 5\pi/12$. And we know $C = E + F$.

So we get $F = C - E = 5\pi/12 - \pi/4 = \pi/6$. And the area t_1 is $\dfrac{2b\sin F}{2}$.

Thus, we get $t_1 = \dfrac{2b\sin F}{2} = 2 \cdot 2\sqrt{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \sqrt{3}$.

So the area of the tetragon $T = t_0 + t_1 = 3 + \sqrt{3} + \sqrt{3} = 3 + 2\sqrt{3}$.