Examples in Trig Algebra

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Examples in Trig Algebra

Note: $(\sin x)(\cos x) = \sin x \cos x$, and $(\sin x)^n = \sin^n x$, so for instance, $(\sin x)^2 = \sin^2 x$.

0. Assuming $\sin x + \cos x = 1/2$, find the values of the expressions below.

$\sin x - \cos x$,
$\sin^3 x + \cos^3 x$,
$\sin^6 x + \cos^6 x$,
$\tan x + \cot x$,
and $\tan^3 x + \cot^3 x$.

1.0. Assuming $\sin x + \cos x = \sin x \cos x = a$, find a and the value of $\tan x + \cot x$.

1.1. Assuming $\sin x + \cos x = b$, and $\sin x \cos x = (b^2 - 2)/4$, find b and the value of $\tan x + \cot x$.

1.2. Assuming $\sin x + \cos x = 0$, find the sum and the product of the two in each pair as follows: $\sin^2 x$ and $\cos^2 x$, $\sin^3 x$ and $\cos^3 x$, and $\tan x$ and $\cot x$.

2.0. Assuming $\sin x = 3/5$ and $\pi/2 < x < \pi$, find $\cos x$ and $\tan x$.

2.1. Assuming $\tan x = -5/12$, find $\sin x$ and $\cos x$.

2.2. Assuming $2\cos x - \sin x = 1$ and $0 < x < \pi/2$, find $\sin x$ and $\cos x$.

2.3. Assuming $6\sin^2 x + \sin x \cos x - 2\cos^2 x = 0$ and $\pi/2 < x < \pi$, find $\sin x + \cos x$, and $\tan x$.

3. Assuming $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{u}$, and $\boldsymbol{v}$ are constants, find in each case below, the connective expression between the constants:

3.0. $\boldsymbol{a}\ \mathbf{sin}\ \boldsymbol{x} - \boldsymbol{b}\ \mathbf{cos}\ \boldsymbol{x} = \mathbf{1}$, and $\boldsymbol{b}\ \mathbf{sin}\ \boldsymbol{x} + \boldsymbol{a}\ \mathbf{cos}\ \boldsymbol{x} = \mathbf{1} + \boldsymbol{b}\ \mathbf{cos}\ \boldsymbol{x}$.

3.1. $\boldsymbol{u} = \boldsymbol{a}\ \mathbf{sin}^{\mathbf{3}}\ \boldsymbol{x}$, and $\boldsymbol{v} = \boldsymbol{b}\ \mathbf{cos}^{\mathbf{3}}\ \boldsymbol{x}$ where $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{b} \neq \mathbf{0}$.

3.2. $\boldsymbol{a}\ \mathbf{sin}\ \boldsymbol{x} + \mathbf{cos}\ \boldsymbol{x} = \mathbf{1}$, and $\boldsymbol{b}\ \mathbf{sin}\ \boldsymbol{x} - \mathbf{cos}\ \boldsymbol{x} = \mathbf{1}$.

3.3. $\boldsymbol{a}\ \mathbf{sec}\ \boldsymbol{x} = \mathbf{1} + \mathbf{tan}\ \boldsymbol{x}$, and $\boldsymbol{b}\ \mathbf{sec}\ \boldsymbol{x} = \mathbf{1} - \mathbf{tan}\ \boldsymbol{x}$.

4. Find the angle between two lines $\boldsymbol{y} = \boldsymbol{ax} + \boldsymbol{b}$ and $\boldsymbol{y} = \boldsymbol{cx} + \boldsymbol{d}$.

Suggestions or Solutions

To the Problem in the Example 0

Assuming $\sin x + \cos x = 1/2$, find the values of the expressions below.
$\sin x - \cos x$, $\sin^3 x + \cos^3 x$, $\sin^6 x + \cos^6 x$, $\tan x + \cot x$, and $\tan^3 x + \cot^3 x$.

Doing algebra, we often do polynomial factorizations. The same is true for trig-algebra, too. And quite often, the trig-identity where $\sin^2 x + \cos^2 x = 1$ is useful.

So for instance, taking the square of the first expression, we get

$(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x = 1 - 2 \sin x \cos x$.

Thus, knowing the value of $\sin x \cos x$, we can get the value of $\sin x - \cos x$.

To begin with, for simplicity, setting $s = \sin x$, and $c = \cos x$, we have $s + c = 1/2$.

So we can get $(s + c)^2 = s^2 + c^2 + 2sc = 1 + 2sc = 1/4$.

Thus, we get $sc = (1/4 - 1)/2 = -3/8$.

So next, beginning with $\sin x - \cos x$, we get

$(s - c)^2 = s^2 + c^2 - 2sc = 1 - 2(-3/8) = 1 + 6/8 = 14/8 = 7/4 \Rightarrow (s - c)^2 = 7/4$.

Thus, we get $\sin x - \cos x = \pm\frac{\sqrt{7}}{2}$.

And using $sc = -3/8$, along with factorization, we get

$s^3 + c^3 = (s + c)^3 - 3sc(s + c) = (1/2)^3 - 3(-3/8)(1/2) = 1/8 + 9/16 = 11/16$.

$s^6 + c^6 = (s^3 + c^3)^2 - 2s^3c^3 = (11/16)^2 - 2(-3/8)^3 = 11^2/2^8 + 27/2^8 = (121+27)/2^8$

$= 148/2^8 = 2\cdot 74/2^8 = 4\cdot 37/2^8 = 37/2^6 = 37/64$.

And we can get the same the way below, too.

$s^6 + c^6 = (s^2 + c^2)^3 - 3s^2c^2(s^2 + c^2) = 1 - 3(-3/8)^2 \cdot 1 = (64 - 27)/64 = 37/64$.

Next, $\tan x + \cot x = s/c + c/s = (s^2 + c^2)/(sc) = 1/(-3/8) = -8/3$.

And next,

$\tan^3 x + \cot^3 x = (\tan x + \cot x)^3 - 3 \tan x \cot x (\tan x + \cot x)$

$= (-8/3)^3 - 3 \cdot 1 \cdot (-8/3) = -2^9/3^3 + 2^3 = (3^3 2^3 - 2^9)/3^3 = (27 \cdot 8 - 16 \cdot 32)/27$

$= (240 - 24 - 320 - 180 - 12)/27 = (-200 + 40 - 124 - 12)/27 = -296/27$.

Suggestions or Solutions

To the Problems in the Example 1

1.0. Assuming sin x + cos x = sin x cos x = a, find a and the value of tan x + cot x.

Assuming first, for simplicity, $s = \sin x$, and $c = \cos x$, we have $s + c = a$, and $sc = a$.

Then, we get $(s + c)^2 = s^2 + c^2 + 2sc \Rightarrow a^2 = 1 + 2a \Rightarrow a^2 - 2a - 1 = 0$.

So next, using the quadratic formula, we get $a = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2}$.

So do we have $a = 1 + \sqrt{2}$ or $1 - \sqrt{2}$?

We have this $-1 \le \sin\theta \le 1$, and $-1 \le \cos\theta \le 1$.

So we get $-1 \le a \le 1$, because $sc = a$. Thus, we get $a = 1 - \sqrt{2}$.

So next, we get $\tan x + \cot x = \dfrac{s}{c} + \dfrac{c}{s} = \dfrac{s^2 + c^2}{sc} = \dfrac{1}{a}$

$$= \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{1+\sqrt{2}}{1-2} = -(1+\sqrt{2}).$$

1.1. assuming sin x + cos x = b, and sin x cos x = $(b^2 - 2)/4$, find b and the value of tan x + cot x.

Setting first, for simplicity, $s = \sin x$, and $c = \cos x$, we get $s + c = b$, and $sc = (b^2 - 2)/4$.
So we get $(s + c)^2 = b^2 = s^2 + c^2 + 2sc = 1 + (b^2 - 2)/2$, since $sc = (b^2 - 2)/4$.
Thus, $b^2 = 1 + (b^2 - 2)/2 \Rightarrow 2b^2 = 2 + b^2 - 2 \Rightarrow b = 0$.
Thus, next, we get $\tan x + \cot x = s/c + c/s = (s^2 + c^2)/(sc) = 1/(sc) = 4/(b^2 - 2) = -2$,
since we have $sc = (b^2 - 2)/4$, and $b = 0$.

1.2. Assuming sin x + cos x = 0, find the sum and the product of the two in each pair as follows: ($\sin^2 x$ and $\cos^2 x$), ($\sin^3 x$ and $\cos^3 x$), and (tan x and cot x).

Setting first, for simplicity, $s = \sin x$, and $c = \cos x$, we have $s + c = 0$, and, we can get

$(s + c)^2 = s^2 + c^2 + 2sc \Rightarrow 0 = 1 + 2sc$, since we have $s^2 + c^2 = 1$, which is a trig-identity, and we are given $s + c = 0$.

Thus, we get $sc = -1/2$.
So beginning with $\sin^2 x$ and $\cos^2 x$, we get $s^2 + c^2 = 1$, and $s^2c^2 = 1/4$.

Next, moving on to $\sin^3 x$ and $\cos^3 x$, we get $s^3 + c^3 = (s + c)^3 - 3sc(s + c) = 0$, since we have $s + c = 0$.

Next, we get $\tan x + \cot x = \frac{s}{c} + \frac{c}{s} = \frac{s^2 + c^2}{cs} = -2$, since we have $sc = -1/2$.

And we get $\tan x \cot x = \frac{s}{c} \cdot \frac{c}{s} = 1$.

Suggestions or Solutions
To the Problems in the Example 2

2.0. Assuming sin $x = 3/5$ and $\pi/2 < x < \pi$, find cos x and tan x.

We know that the sine is the ratio of the opposite to the hypotenuse in a right triangle.

In short, **sin x** is the opposite over the hypotenuse.

And we know that the opposite faces the angle x, because x is the governing angle.

So from **sin $x = 3/5$**, we can see that the ratio of the opposite to the hypotenuse is **3/5**.

Thus, assuming O is the opposite, H is the hypotenuse, and A is the adjacent, we get $O/H = 3/5$.

So assuming $O = 3a$ where a is a constant > 0, we get $H = 5a$.

What then about the adjacent A?

We know that O, H, and A are three sides in a right triangle.

So we can get A using the distance formula, called Pythagorean theorems, too.
Thus, using the formula, we get

$$H^2 = O^2 + A^2 \Rightarrow (5a)^2 = (3a)^2 + A^2 \Rightarrow A^2 = (25 - 9)a = 16a \Rightarrow A = 4a.$$

And we know that the cosine is the adjacent over the hypotenuse, and that the tangent is the opposite over the adjacent.

So we get $\cos x = A/H = 4a/5a = 4/5$, and $\tan x = O/A = 3a/4a = 3/4$.

We have however, $\pi/2 < x < \pi$. So we get $-1 < \cos x < 0$, and $\tan x < 0$.
Thus, we get $\cos x = -4/5$, and $\tan x = -3/4$.
Note that in this case, the vector is in the second quadrant, and in the second quadrant, the sine only is positive.

2.1. Assuming tan x = -5/12, find sin x and cos x.

From **tan x = -5/12**, we can see that the opposite over the adjacent = -5/12, which is negative. So the vector is either in the second quadrant or in the fourth quadrant, since the tangent is negative there.

And taking the absolute value of the **tan** above, we get $|\tan x| = |-5/12| = 5/12$,

Thus, assuming O is the opposite, H is the hypotenuse, and A is the adjacent, we get $|O/A| = 5/12$.

So assuming $O = 5a$ where a is a constant > 0, we get $A = 12a$.

And we can get H using the distance formula. So using the formula, we get

$$H^2 = O^2 + A^2 = (5a)^2 + (12a)^2 \Rightarrow H^2 = (25 + 144)a = 169a \Rightarrow H = 13a.$$

And we know that the sine is the opposite over the hypotenuse, and that the cosine is the adjacent over the hypotenuse.
So we can set for now

$|\sin x| = |O/H| = 5a/13a = 5/13$, and $|\cos x| = |O/A| = 12a/13a = 12/13$.

And we know that if $\tan x < 0$, the vector is in the second or the fourth quadrant.

Also, we know if the vector is in the second quadrant, the sine is positive, but the cosine is negative. And if the vector is in the fourth quadrant, the sine is negative but the cosine is positive.

So given $\tan x = -5/12$, assuming the vector is in the second quadrant,
we get $\sin x = 5/13$ and $\cos x = -12/13$,
or assuming the vector is in the fourth quadrant, we get $\sin x = -5/13$ and $\cos x = 12/13$.

2.2. Assuming $2\cos x - \sin x = 1$ and $0 < x < \pi/2$, find $\sin x$ and $\cos x$.

Using the equation given, we can get an equation for **cos *x***. Solving the equation then, we can get the value of **cos *x***. How then can we get the equation?

We have a trig-identity $\sin^2 x + \cos^2 x = 1$. And we have this, too: $2\cos x - \sin x = 1$.

So we can put the identity in terms of **cos *x***, and then, we get the equation for **cos *x***.
To begin with, we get $2\cos x - \sin x = 1 \Rightarrow \sin x = 2\cos x - 1$.

So we get $\sin^2 x + \cos^2 x = (2\cos x - 1)^2 + \cos^2 x = 4\cos^2 x - 4\cos x + 1 + \cos^2 x$

$= 5\cos^2 x - 4\cos x + 1 = 1 \Rightarrow \cos x\,(5\cos x - 4) = 0 \Rightarrow \cos x = 0$ or $4/5$.

And we have this, too: $0 < x < \pi/2$. So we get $\cos x = 4/5$.
What then about **sin *x***?

We have $\sin x = 2\cos x - 1$. So we get $\sin x = 2(4/5) - 1 = 3/5$.
What then about **tan *x***?

We know $\tan x = \sin x / \cos x$. So we get $\tan x = (3/5)/(4/5) = \dfrac{\frac{3}{5}}{\frac{4}{5}} = 3/4$.

By the way, we have this, too: $\dfrac{\frac{3}{5}}{\frac{4}{5}} = \dfrac{3}{5}\cdot\dfrac{5}{4} = \dfrac{3}{4}$

2.3. Assuming $6\sin^2 x + \sin x \cos x - 2\cos^2 x = 0$ and $\pi/2 < x < \pi$, find $\sin x + \cos x$, and $\tan x$.

Examining the equation give, we can modify it into an equation for **tan *x***.

Solving the equation then we can get the value of **tan *x***.

So to begin with, dividing by $\mathbf{\cos^2} \boldsymbol{x}$ the equation $\mathbf{6\sin^2}\, \boldsymbol{x} + \mathbf{\sin}\, \boldsymbol{x}\, \mathbf{\cos}\, \boldsymbol{x} - \mathbf{2\cos^2}\, \boldsymbol{x} = \mathbf{0}$, we get $\mathbf{6\tan^2}\, \boldsymbol{x} + \mathbf{\tan}\, \boldsymbol{x} - \mathbf{2} = \mathbf{0}$.

And next, factoring it, we can get $\mathbf{(3\tan}\, \boldsymbol{x} + \mathbf{2)(2\tan}\, \boldsymbol{x} - \mathbf{1)} = \mathbf{0}$.

So we get $\mathbf{\tan}\, \boldsymbol{x} = \mathbf{-2/3}$ or $\mathbf{1/2}$.

And we have this too: $\pi/2 < x < \pi$. That is, the vector is in the second quadrant, so the tangent is negative..

Thus, we can see that $\mathbf{\tan}\, \boldsymbol{x} = \mathbf{-2/3}$. What then about $\mathbf{\sin}\, \boldsymbol{x} + \mathbf{\cos}\, \boldsymbol{x}$?

We know that the tangent is the opposite over the hypotenuse:

And we can get this: $|\tan x| = |-2/3| = 2/3$.

So we get **|the opposite| = 2**, and **|the adjacent| = 3**.

And by the distance formula, we get **the hypotenuse** $= \sqrt{2^2 + 3^2} = \sqrt{13}$.

So we get $|\sin x| = \frac{2}{\sqrt{13}}$, and $|\cos x| = \frac{3}{\sqrt{13}}$. And we have $\pi/2 < x < \pi$.

So the vector is in the second quadrant. The sine is positive, but the cosine is negative there. Thus, we get $\sin x + \cos x = \frac{2}{\sqrt{13}} - \frac{3}{\sqrt{13}} = -\frac{1}{\sqrt{13}} = -\frac{\sqrt{13}}{13}$.

By the way, solving a trig-equation $\sin x = 1$ with no more specification, we don't want to put the solution this way: $x = \pi/2$.

That's because it's not the only solution.

We can get $\sin x = 1$ when not only $x = \pi/2$ but $x = 2\pi + \pi/2$, $-\pi - \pi/2$, $-\pi/2$, etc., too.

And in fact, we can get $\sin x = 1$ when $x = n\pi + (-1)^n(\pi/2)$ for n integer.

Also, the same is true for other ratios. So for instance,

$\sin x = 1/2 \Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$ for n integer.

$\sin 2x = 1/2 \Rightarrow 2x = n\pi + (-1)^n \frac{\pi}{6}$ for n integer $\Rightarrow x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$ for n integer.

$\sin x = -1/2 \Rightarrow x = n\pi + (-1)^n(-\frac{\pi}{6})$ for n integer $\Rightarrow x = n\pi - (-1)^n \frac{\pi}{6}$ for n integer.

$\mathbf{cos}\ \boldsymbol{x} = \mathbf{1/2} \Rightarrow \boldsymbol{x} = \mathbf{2}\boldsymbol{n}\pi \pm \frac{\pi}{3}$ for $\boldsymbol{n}$ integer.

$\mathbf{tan}\ \boldsymbol{x} = \frac{\sqrt{\mathbf{2}}}{\mathbf{2}} \Rightarrow \boldsymbol{x} = \boldsymbol{n}\pi + \frac{\pi}{4}$ for $\boldsymbol{n}$ integer.

That is to say that

$\mathbf{sin}\ \boldsymbol{x} = \mathbf{sin}\ \boldsymbol{a} \Rightarrow \boldsymbol{x} = \boldsymbol{n}\pi + \mathbf{(-1)}^{\boldsymbol{n}}\boldsymbol{a}$ for $\boldsymbol{n}$ integer.

$\mathbf{cos}\ \boldsymbol{x} = \mathbf{cos}\ \boldsymbol{b} \Rightarrow \boldsymbol{x} = \mathbf{2}\boldsymbol{n}\pi \pm \boldsymbol{b}$ for $\boldsymbol{n}$ integer.

$\mathbf{tan}\ \boldsymbol{x} = \mathbf{tan}\ \boldsymbol{c} \Rightarrow \boldsymbol{x} = \boldsymbol{n}\pi + \boldsymbol{c}$ for $\boldsymbol{n}$ integer.

Suggestions or Solutions
To the Problems in the Example 3

Assuming a, b, u, and v are constants, find in each case below, the connective expression between the constants.

3.0. $a \sin x - b \cos x = 1$, and $b \sin x + a \cos x = 1 + b \cos x$.

3.1. $u = a \sin^3 x$, and $v = b \cos^3 x$ where $a \neq 0$ and $b \neq 0$.

3.2. $a \sin x + \cos x = 1$, and $b \sin x - \cos x = 1$.

3.3. $a \sec x = 1 + \tan x$, and $b \sec x = 1 - \tan x$.

3.0. Assuming $s = \sin x$, and $c = \cos x$,
we get $as - bc = 1$…(1), and $bs + ac = 1 + bc$… (2) Let's next, remove c first.

Multiplying (1) by $(a - b)$, we get $as(a - b) - bc(a - b) = 1(a - b) = a - b$.
Next, multiplying (2) by b, we get $b^2s + abc = b + b^2c \Rightarrow b^2s + bc(a - b) = b$.

So we get $as(a - b) - bc(a - b) = a - b$, and $b^2s + bc(a - b) = b$.

Next, adding together the two equations above, we get
$as(a - b) + b^2s = a - b + b \Rightarrow (a^2 - ab + b^2)s = a$.

Next, removing s from the two equations (1) and (2), we can do it the way below.

Multiplying (1) by b, we get $(as - bc)b = 1b \Rightarrow abs - b^2c = b$ …(3)
Multiplying (2) by a, we get $(bs + ac)a = (1 + bc)a \Rightarrow abs + a^2c = a + abc$ …(4)

So next, subtracting (4) from (3), we get
$-b^2c - a^2c = b - a - abc \Rightarrow a^2c + b^2c = a - b + abc \Rightarrow (a^2 - ab + b^2)c = a - b$.

What then is the next?

We have: $s^2 + c^2 = 1$, and the two equations as follows:
$(a^2 - ab + b^2)s = a$, and $(a^2 - ab + b^2)c = a - b$.

So we can get the connective expression we want the way below.

Solving first, for s the equation $(a^2 - ab + b^2)s = a$, we get the expression for s.
Also, solving for c the equation $(a^2 - ab + b^2)c = a - b$, we get the expression for c.

And next, putting into the identity $s^2 + c^2 = 1$, the expression for s and the expression for c, we can get the connective expression we want.

We want to make sure first though, $a^2 - ab + b^2$ is not 0. So checking it, we get:

$$a^2 - ab + b^2 = a^2 - ab + b^2/4 - b^2/4 + b^2 = (a - b/2)^2 + 3b^2/4 > 0.$$

That's because a and b cannot be 0 at the same time due to the equation (1) above.
Thus, we can get

$$(a^2 - ab + b^2)s = a \Rightarrow s = \frac{a}{a^2 - ab + b^2}.$$

$$(a^2 - ab + b^2)c = a - b \Rightarrow c = \frac{a - b}{a^2 - ab + b^2}.$$

So next, putting the s and the c above into the identity $s^2 + c^2 = 1$, we get:

$$\left(\frac{a}{a^2 - ab + b^2}\right)^2 + \left(\frac{a - b}{a^2 - ab + b^2}\right)^2 = \frac{a^2 + (a - b)^2}{(a^2 - ab + b^2)^2} = \frac{2a^2 - 2ab + b^2}{(a^2 - ab + b^2)^2} = 1.$$

So we get

$$2a^2 - 2ab + b^2 = (a^2 - ab + b^2)^2 \Rightarrow (a^2 - ab + b^2) + a^2 - ab = (a^2 - ab + b^2)^2$$

$$(a^2 - ab + b^2)\{(a^2 - ab + b^2) - 1 - a^2 + ab\} = (a^2 - ab + b^2)(b^2 - 1) = 0.$$

Thus, the connective expression is $(a^2 - ab + b^2)(b^2 - 1) = 0$.

3.1. $u = a \sin^3 x$**, and** $v = b \cos^3 x$ **where** $a \neq 0$ **and** $b \neq 0$**.** To begin with, we can get

$u = a \sin^3 x \Rightarrow \sin x = (u/a)^{1/3} = \sqrt[3]{\frac{u}{a}}$, and $v = b \cos^3 x \Rightarrow \cos x = (v/b)^{1/3} = \sqrt[3]{\frac{v}{b}}$.

So next, we get $\sin^2 x + \cos^2 x = (u/a)^{2/3} + (v/b)^{2/3} = 1$.

Thus, the connective expression is $\sqrt[3]{\frac{u^2}{a^2}} + \sqrt[3]{\frac{v^2}{b^2}} = 1$.

3.2. $a \sin x + \cos x = 1$**, and** $b \sin x - \cos x = 1$**.**

Assuming first, $s = \sin x$, and $c = \cos x$, we get $as + c = 1$ … (1), and $bs - c = 1$ … (2).
So next, we can get
$(1) + (2) \Rightarrow (a + b)s = 2 \Rightarrow s = 2/(a + b)$.
$(1) \cdot b - (2) \cdot a = bc + ac = b - a \Rightarrow c = (b - a)/(a + b)$.

Thus, next, we can get
$s^2 + c^2 = 1 \Rightarrow 4/(a + b)^2 + (b - a)^2/(a + b)^2 = \{4 + (b - a)^2\}/(a + b)^2 = 1$
$\Rightarrow 4 + b^2 - 2ab + a^2 = a^2 + 2ab + b^2 \Rightarrow 4ab = 4 \Rightarrow ab = 1$, which is the connective expression.

3.3. $a \sec x = 1 + \tan x$**, and** $b \sec x = 1 - \tan x$**.**

Assuming first, $s = \sin x$, and $c = \cos x$, we get
$a/c = 1 + s/c \Rightarrow a = c + s$ … (1), and $b/c = 1 - s/c \Rightarrow b = c - s$ … (2).

So we get:
(1) + (2) $\Rightarrow a + b = 2c \Rightarrow c = (a + b)/2$, and (1) – (2) $\Rightarrow a - b = 2s \Rightarrow s = (a - b)/2$.

Thus, we get $s^2 + c^2 = 1 \Rightarrow (a - b)^2/4 + (a + b)^2/4 = (a^2 - 2ab + b^2 + a^2 + 2ab + b^2)/4$

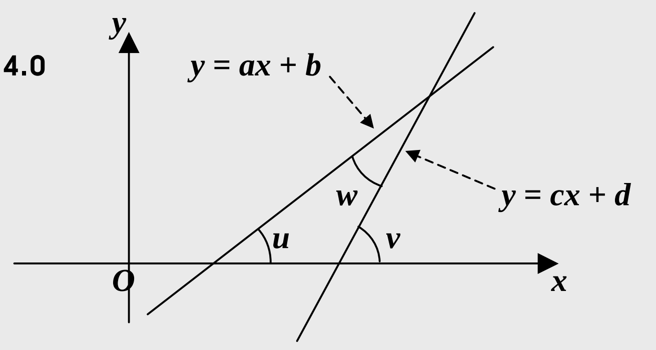
$= 2(a^2 + b^2)/4 = (a^2 + b^2)/2 = 1 \Rightarrow a^2 + b^2 = 2$, which is the connective expression.

Suggestions or Solutions
To the Problem in the Example 4

Find the angle between two lines $y = ax + b$ and $y = cx + d$.

Suppose first, $\tan u = a$, $\tan v = c$, and $0 < a < c$.
Then, putting the two lines in the x-y plane, we can put them the way below.

Fig. 4.0



And we know that $v = u + w$, because $u + v + w = \pi$.

That is to say that we have $w = v - u$.

Then, we get $\tan w = \tan (v - u)$.

And we have $\tan (v \pm u) = \dfrac{\tan v \pm \tan u}{1 \mp \tan v \tan u}$.

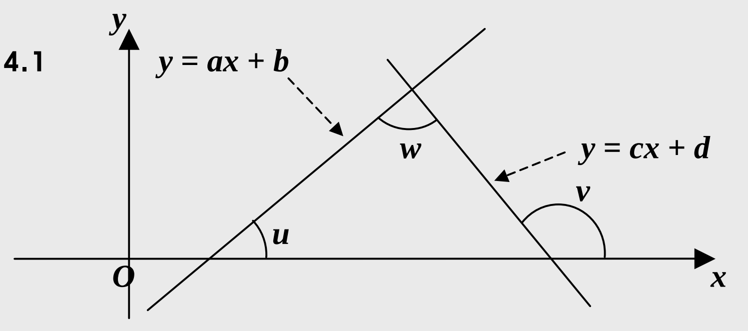
So we get $\tan w = \tan (v - u) = \dfrac{\tan v - \tan u}{1 + \tan v \tan u} = \dfrac{c - a}{1 + ca}$.

Thus, we get $w = \tan^{-1} \dfrac{c - a}{1 + ca}$.

Suppose next, $\tan u = a$, $\tan v = c$, $a > 0$, and $c < 0$.

Then, putting the two lines in the x-y plane, we can put them the way below.

Fig. 4.1



Then again, since $v = u + w$, we get $w = v - u$, too. So we get $\tan w = \tan(v - u)$.

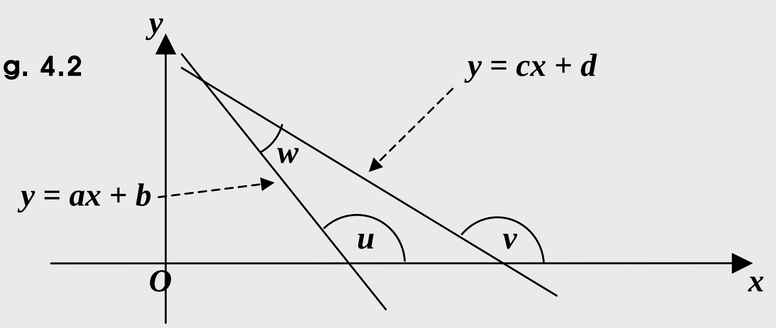
Thus, we get $\tan w = \tan(v - u) = \dfrac{\tan v - \tan u}{1 + \tan v \tan u} = \dfrac{c - a}{1 + ca}$.

So we get $w = \tan^{-1} \dfrac{c - a}{1 + ca}$.

Suppose next, $\tan u = a$, $\tan v = c$, and $a < c < 0$.

Then, putting the two lines in the x-y plane, we can put them the way below.

Fig. 4.2



Then again, since $v = u + w$, we get $w = v - u$, too. So we get $\tan w = \tan(v - u)$.

Thus, we get $\tan w = \tan(v - u) = \dfrac{\tan v - \tan u}{1 + \tan v \tan u} = \dfrac{c - a}{1 + ca}$.

So we get $w = \tan^{-1} \dfrac{c - a}{1 + ca}$.

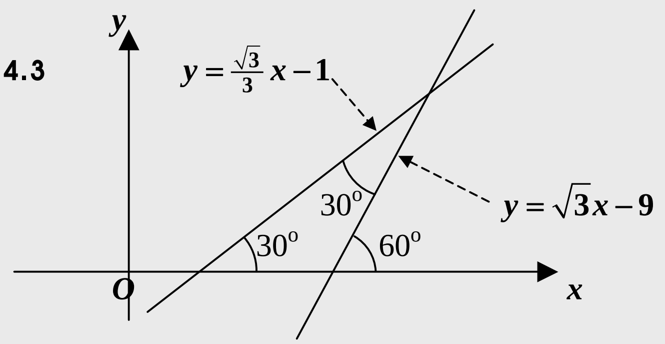
So we can say that assuming $\boldsymbol{w}$ is the angle between two lines $\boldsymbol{y = ax + b}$ and $\boldsymbol{y = cx + d}$, we get $\boldsymbol{w = \tan^{-1}\frac{c-a}{1+ca}}$.

Suppose for instance, $\boldsymbol{w}$ is the angle between two lines $\boldsymbol{y = \frac{\sqrt{3}}{3}x - 1}$ and $\boldsymbol{y = \sqrt{3}x - 9}$.

Then, $\boldsymbol{\tan \pi/6 = \frac{\sqrt{3}}{3} = a}$, and $\boldsymbol{\tan \pi/3 = \sqrt{3} = c}$. That is, $\boldsymbol{u = \pi/6}$, and $\boldsymbol{v = \pi/3}$. So $\boldsymbol{w = \pi/6}$.

And putting the two lines in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, we can put them the way below.

Fig. 4.3



Then, we get $\boldsymbol{\frac{c-a}{1+ca} = \frac{\sqrt{3}-\frac{\sqrt{3}}{3}}{1+\sqrt{3}\cdot\frac{\sqrt{3}}{3}} = \frac{\frac{3\sqrt{3}-\sqrt{3}}{3}}{2} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}}$, the arc tan of which is $\boldsymbol{\pi/6}$.

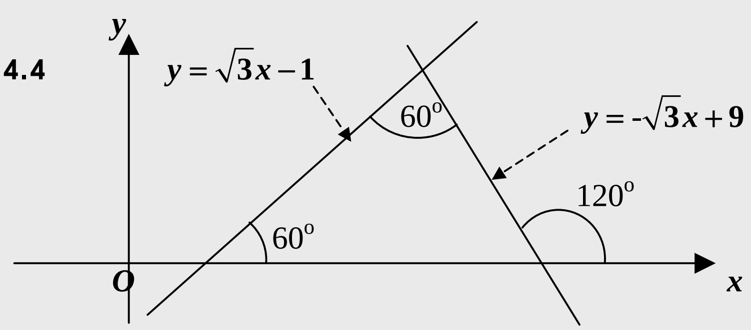
Suppose next, $\boldsymbol{w}$ is the angle between two lines $\boldsymbol{y = \sqrt{3}x - 1}$ and $\boldsymbol{y = -\sqrt{3}x + 9}$.

Then, $\boldsymbol{\tan \pi/3 = \sqrt{3} = a}$, and $\boldsymbol{\tan 2\pi/3 = \tan(\pi - \pi/3) = -\tan \pi/3 = -\sqrt{3} = c}$.

That is, $\boldsymbol{u = \pi/3}$, and $\boldsymbol{v = 2\pi/3}$. So $\boldsymbol{w = \pi/3}$.

And putting the two lines in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, we can put them the way below.

Fig. 4.4



Then, we get $\frac{c-a}{1+ca} = \frac{-\sqrt{3}-\sqrt{3}}{1+(-\sqrt{3})\cdot\sqrt{3}} = \frac{-2\sqrt{3}}{1-3} = \sqrt{3}$, the arc tan of which is **π/3**.

Suppose next, w is the angle between two lines $y = \frac{\sqrt{3}}{3}x - 1$ and $y = -\sqrt{3}x + 9$.

Then, **tan π/6** = $\frac{\sqrt{3}}{3} = a$, and **tan 2π/3 = tan(π – π/3) = -tan π/3 = -$\sqrt{3}$** $= c$.

That is, u = **π/6**, and v = **2π/3**. So w = **π/2**.

And putting the two lines in the ***x-y*** plane, we can put them the way below.

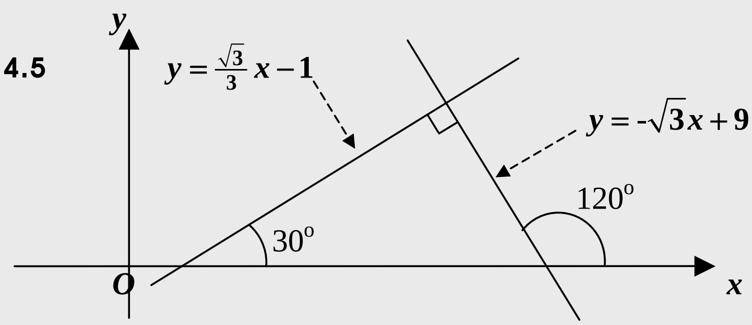


Fig. 4.5

Then, we get $\frac{c-a}{1+ca} = \frac{-\sqrt{3}-\frac{\sqrt{3}}{3}}{1+(-\sqrt{3})\cdot\frac{\sqrt{3}}{3}} = \frac{\frac{-3\sqrt{3}-\sqrt{3}}{3}}{0}$, which means w = **π/2**.

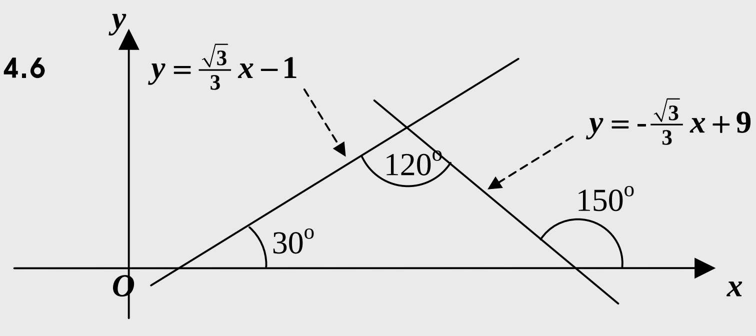
Suppose next, w is the angle between two lines $y = \frac{\sqrt{3}}{3}x - 1$ and $y = -\frac{\sqrt{3}}{3}x + 9$.

Then, **tan π/6** = $\frac{\sqrt{3}}{3} = a$, and **tan 5π/6 = tan(π – π/6) = -tan π/6 =** $-\frac{\sqrt{3}}{3} = c$.

That is, u = **π/6**, and v = **5π/6**. So w = **2π/3**.

And putting the two lines in the ***x-y*** plane, we can put them the way below.

Fig. 4.6



Then, we get $\dfrac{c-a}{1+ca} = \dfrac{-\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}}{1+(-\frac{\sqrt{3}}{3})\cdot\frac{\sqrt{3}}{3}} = \dfrac{-\frac{2\sqrt{3}}{3}}{1-\frac{1}{3}} = \dfrac{-\frac{2\sqrt{3}}{3}}{\frac{2}{3}} = -\dfrac{2\sqrt{3}}{2} = -\sqrt{3}$,

the arc tan of which is **2π/3**.

Suppose next, ***w*** is the angle between two lines $y = -\sqrt{3}x + 5$ and $y = -\frac{\sqrt{3}}{3}x + 4$.

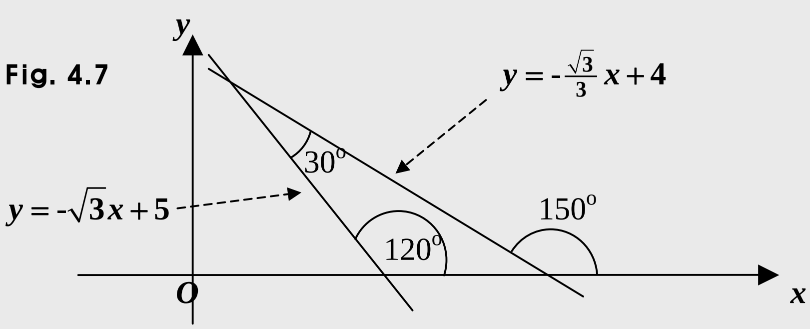
Then, **tan 2π/3 = tan(π – π/3) = -tan π/3 =** $-\sqrt{3} = a$, and

tan 5π/6 = tan (π – π/6) = -tan π/6 = $-\frac{\sqrt{3}}{3} = c$.

That is, ***u*** **= 2π/3**, and ***v*** **= 5π/6**. So ***w*** **= π/6**.

And putting the two lines in the ***x-y*** plane, we can put them the way below.

Fig. 4.7



Then, we get $\dfrac{c-a}{1+ca} = \dfrac{-\frac{\sqrt{3}}{3} - (-\sqrt{3})}{1+(-\frac{\sqrt{3}}{3})\cdot(-\sqrt{3})} = \dfrac{-\frac{\sqrt{3}}{3} + \sqrt{3}}{1+\frac{\sqrt{3}}{3}\cdot\sqrt{3}} = \dfrac{\frac{-\sqrt{3}+3\sqrt{3}}{3}}{2} = \dfrac{2\sqrt{3}}{6} = \dfrac{\sqrt{3}}{3}$, the arc tan

of which is **π/6**.